# Stochastic Optimization
# Decomposition Methods for Two-stage problems

V. Leclère

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École des Ponts ParisTech

# Presentation Outline

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# Two-stage Problem

The extensive formulation of

$$\begin{aligned}\min_{u_0,\boldsymbol{u}_1} \quad & \mathbb{E}\Big[L(u_0,\boldsymbol{\xi},\boldsymbol{u}_1)\Big] \\ s.t. \quad & g(u_0,\boldsymbol{\xi},\boldsymbol{u}_1) \leq 0, \qquad \mathbb{P}-a.s \\ & \sigma(\boldsymbol{u}_1) \subset \sigma(\boldsymbol{\xi})\end{aligned}$$

is

$$\begin{aligned}\min_{u_0,\{u_1^s\}_{s\in[\![1,S]\!]}} \quad & \sum_{s=1}^{S} \pi^s L(u_0,\xi^s,u_1^s) \\ s.t \quad & g(u_0,\xi^s,u_1^s) \leq 0, \qquad \forall s \in [\![1,S]\!].\end{aligned}$$

It is a deterministic problem that can be solved with standard tools or specific methods.

# Splitting variables

The extended Formulation (in a compact formulation)

$$\begin{aligned} \min_{u_0, \{u_1^s\}_{s\in[\![1,S]\!]}} \quad & \sum_{s=1}^{S} \pi^s L(u_0^s, \xi^s, u_1^s) \\ s.t \quad & g(u_0, \xi^s, u_1^s) \leq 0, \qquad \forall s \in [\![1, S]\!]. \end{aligned}$$

Can be written in a splitted formulation

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Can be written in a splitted formulation

$$\begin{aligned} \min_{\bar{u}_0\{u_0^s,u_1^s\}_{s\in[\![1,S]\!]}} \quad & \sum_{s=1}^{S} \pi^s L(u_0^s, \xi^s, u_1^s) \\ s.t \quad & g(u_0^s, \xi^s, u_1^s) \leq 0, \qquad \forall s \in [\![1, S]\!] \\ & u_0^s = \bar{u}_0 \end{aligned}$$

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& u_0^s = \sum_{s'} \pi^{s'} u_0^{s'} && \forall s
\end{aligned}
$$

# Dualizing non-anticipativity constraint I

$$\min_{\{u_0^s,u_1^s\}_{s\in[\![1,S]\!]}} \quad \sum_{s=1}^{S} \pi^s L(u_0^s, \xi^s, u_1^s)$$
$$s.t \quad g(u_0^s, \xi^s, u_1^s) \leq 0, \qquad \forall s \in [\![1, S]\!]$$
$$u_0^s = \sum_{s'} \pi^{s'} u_0^{s'} \qquad \forall s$$

is equivalent to

$$\min_{\{u_0^s,u_1^s\}_{s\in[\![1,S]\!]}} \max_{\{\lambda^s\}_{s\in[\![1,S]\!]}} \quad \sum_{s=1}^{S} \pi^s L(u_0^s, \xi^s, u_1^s) + \pi^s \lambda^s \Big( u_0^s - \sum_{s'} \pi^{s'} u_0^{s'} \Big)$$

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& + \sum_{s=1}^{S} \pi^s \lambda^s u_0^s - \sum_{s,s'} \pi^s \lambda^s \pi^{s'} u_0^{s'} \\
s.t \quad & g(u_0^s, \xi^s, u_1^s) \le 0, \qquad \forall s \in [\![1, S]\!]
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& + \sum_{s=1}^{S} \pi^s \Big(\lambda^s - \mathbb{E}\big[\boldsymbol{\lambda}\big]\Big) u_0^s \\
s.t \quad & g(u_0^s, \xi^s, u_1^s) \leq 0, \qquad \forall s \in [\![1, S]\!]
\end{aligned}
$$

## Dualizing non-anticipativity constraint II

Thus, the dual problem reads

$$\max_{\boldsymbol{\lambda}} \quad \min_{\{u_0^s, u_1^s\}_{s\in[\![1,S]\!]}} \quad \sum_{s=1}^{S} \pi^s \Big( L(u_0^s, \xi^s, u_1^s) + \Big(\lambda^s - \mathbb{E}\big[\boldsymbol{\lambda}\big]\Big) u_0^s \Big)$$

$$s.t \quad g(u_0^s, \xi^s, u_1^s) \le 0, \qquad \forall s \in [\![1, S]\!]$$

The inner minimization problem, for $\boldsymbol{\lambda}$ given, can decompose scenario by scenario, by solving $S$ deterministic problem

$$\min_{\{u_0^s, u_1^s\}} \quad L(u_0^s, \xi^s, u_1^s) + \lambda^s u_0^s$$

$$s.t \qquad g(u_0^s, \xi^s, u_1^s) \le 0$$

# Dualizing non-anticipativity constraint II

Thus, the dual problem reads

$$\max_{\boldsymbol{\lambda}:\mathbb{E}[\boldsymbol{\lambda}]=0} \min_{\{u_0^s,u_1^s\}_{s\in[\![1,S]\!]}} \quad \sum_{s=1}^{S} \pi^s \Big( L(u_0^s, \xi^s, u_1^s) + \Big(\lambda^s \qquad\qquad \Big) u_0^s \Big)$$

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$$s.t \qquad g(u_0^s, \xi^s, u_1^s) \leq 0$$

## Price of information

- By weak duality, any $\boldsymbol{\lambda}$ such that $\mathbb{E}\big[\boldsymbol{\lambda}\big] = 0$ will give a lower bound on the 2-stage problem, computed as

$$\sum_{s=1}^{S} \pi^s \quad \min_{u_0^s, u_1^s} \quad \Big( L(u_0^s, \xi^s, u_1^s) + \lambda^s u_0^s \Big)$$
$$s.t \quad g(u_0^s, \xi^s, u_1^s) \leq 0$$

- $\boldsymbol{\lambda} = 0$ lead to the anticipative lower-bound
- If problem is convex, and under some qualification assumptions, there exists an optimal $\boldsymbol{\lambda}^*$, called the price of information, such that the lower bound is tight.

# Progressive Hedging Algorithm

The progressive hedging algorithm build on this decomposition in the following way.

1. Set a price of information $\{\lambda^s\}_{s\in[\![1,S]\!]}$ such that $\mathbb{E}\big[\boldsymbol{\lambda}\big] = 0$
2. For each scenario solve

$$\begin{aligned} \min_{u_0^s, u_1^s} \quad & L(u_0^s, \xi^s, u_1^s) + \lambda^s u_0^s \\ s.t \quad & g(u_0^s, \xi^s, u_1^s) \leq 0 \end{aligned}$$

3. Compute the mean first control $\bar{u}_0 := \sum_{s=1}^S \pi^s u_0^s$
4. Update the price of information with

$$\lambda^s := \lambda^s + \rho(u_0^s - \bar{u}_0)$$

5. Go back to 2.

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$$\begin{aligned} \min_{u_0^s, u_1^s} \quad & L(u_0^s, \xi^s, u_1^s) + \lambda^s u_0^s + \rho \| u_0^s - \bar{u}_0 \|^2 \\ s.t \quad & g(u_0^s, \xi^s, u_1^s) \leq 0 \end{aligned}$$

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# Convergence of Progressive Hedging

**Theorem**

*Assume that $L$ and $g$ are convex lsc in $(u_0, u_1)$ for all $\xi$, and that, for all $s \in S$, there exists $(u_0^s, u_1^s)$ such that $L(u_0^s, \xi^s, u_1^s) < +\infty$ and $g(u_0^s, \xi^s, u_1^s) < 0$.*
*Then, the progressive hedging algorithm converges toward an optimal primal solution, and the price of information converges toward an optimal price of information.*

Moreover we can show that

$$\varepsilon_k = \sqrt{\|(u_0^k, u_1^k) - (u_0^\sharp, u_1^\sharp)\|_2^2 + \frac{1}{\rho^2}\|\lambda - \lambda^\sharp\|_2^2},$$

is a decreasing sequence.

# Bounds in Progressive Hedging

- At any iteration of the PH algorithm, we have a collection of primal solution $\{(u_0^s, u_1^s)\}_{s\in S}$, and a price of information $\{\lambda^s\}_{s\in S}$.
- We have a lower bound on the value of the stochastic programm given by

$$LB^{PH} = \sum_{s\in S} \pi^s \big[L(u_0^s, \xi^s, u_1^s) + \lambda^s u_0^s\big],$$

- and an upper bound given by

$$UB^{PH} = \sum_{s\in S} \pi^s L(\bar{u}_0, \xi^s, u_1^s(\bar{u}_0)).$$

where $u_1^s(\bar{u}_0)$ is the optimal recourse for the first-stage control $\bar{u}_0$

# Presentation Outline

1. Lagrangian decomposition
2. L-Shaped decomposition method
3. Multistage program

# Linear 2-stage stochastic program

Consider the following problem

$$\begin{aligned}
\min \quad & \mathbb{E}\Big[c^\top u_0 + \boldsymbol{q}^\top \boldsymbol{u}_1\Big] \\
s.t. \quad & Au_0 = b, \quad u_0 \geq 0 \\
& \boldsymbol{T}u_0 + \boldsymbol{W}\boldsymbol{u}_1 = \boldsymbol{h}, \quad \boldsymbol{u}_1 \geq 0, \qquad \mathbb{P}-a.s. \\
& u_0 \in \mathbb{R}^n, \quad \sigma(\boldsymbol{u}_1) \subset \sigma(\underbrace{\boldsymbol{q}, \boldsymbol{T}, \boldsymbol{W}, \boldsymbol{h}}_{\boldsymbol{\xi}})
\end{aligned}$$

Which we rewrite

$$\begin{aligned}
\min_{u_0 \geq 0} \quad & c^\top u_0 + \mathbb{E}\Big[Q(u_0, \xi)\Big] \\
s.t. \quad & Au_0 = b
\end{aligned}$$

with

$$\begin{aligned}
Q(u_0, \xi) := \min_{u_1 \geq 0} \quad & q_\xi^\top u_1 \\
s.t. \quad & W_\xi u_1 = h_\xi - T_\xi u_0
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## Linear 2-stage stochastic program : Extensive Formulation

The associated extensive formulation read

$$\begin{aligned} \min \quad & c^\top u_0 + \sum_{s=1}^{S} \pi^s \, q^s \cdot u_1^s \\ s.t. \quad & A u_0 = b, \quad u_0 \geq 0 \\ & T^s u_0 + W^s u_1^s = h^s, \quad u_1^s \geq 0, \forall s \end{aligned}$$

Which we rewrite

$$\begin{aligned} \min_{u_0} \quad & c^\top u_0 + \sum_{s=1}^{S} \pi^s Q^s(u_0) \\ s.t. \quad & A u_0 = b, \quad u_0 \geq 0 \end{aligned}$$

with

$$\begin{aligned} Q^s(u_0) := \min_{u_1 \geq 0} \quad & q^s \cdot u_1 \\ s.t. \quad & W^s u_1 = h^s - T^s u_0 \end{aligned}$$

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# Relatively complete recourse

We assume here relatively complete recourse. Without this assumption we would need feasability cuts.
Here, relatively complete recourse means that, for $u_0 \geq 0$ :

$$Au_0 = b \quad \Longrightarrow \quad Q_s(u_0) < +\infty, \quad \forall s \in [\![1, S]\!]$$

# Decomposition of linear 2-stage stochastic program

We rewrite the extended formulation as

$$\begin{aligned} \min_{u_0, (\theta^s)_{s \in S}} \quad & c^\top u_0 + \sum_s \pi^s \theta^s \\ s.t. \quad & A u_0 = b, \quad u_0 \geq 0 \\ & \theta^s \geq Q^s(u_0) & \forall s \end{aligned}$$

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s.t. \quad & A u_0 = b, \quad u_0 \geq 0 \\
& \theta^s \geq \alpha_k^s \cdot u_0 + \beta_k^s \qquad \forall k, \forall s
\end{aligned}
$$

Note that $Q^s(u_0)$ is a polyhedral function of $u_0$, hence $\theta^s \geq Q^s(u_0)$ can be rewritten $\theta^s \geq \alpha_k^s \cdot u_0 + \beta_k^s, \forall k$.

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Note that $Q^s(u_0)$ is a polyhedral function of $u_0$, hence $\theta^s \geq Q^s(u_0)$ can be rewritten $\theta^s \geq \alpha_k^s \cdot u_0 + \beta_k^s, \forall k$.
The decomposition approach consists in constructing iteratively cut coefficients $\alpha_k^s$ and $\beta_k^s$.

# Obtaining (optimality) cuts I

Recall that

$$
\begin{aligned}
Q^s(u_0) := \min_{u_1^s \in \mathbb{R}^n} \quad & q^s \cdot u_1^s \\
s.t. \quad & W^s u_1^s = h^s - T^s u_0, \quad u_1^s \geq 0
\end{aligned}
$$

can also be written (through strong duality by relatively complete recourse assumption)

$$
\begin{aligned}
(D_{u_0}) \quad Q^s(u_0) = \max_{\lambda^s \in \mathbb{R}^m} \quad & \lambda^s \cdot \left(h^s - T^s u_0\right) \\
s.t. \quad & (W^s)^\top \lambda^s \leq q^s
\end{aligned}
$$

# Obtaining (optimality) cuts II

$$
(D_{u_0}) \quad Q^s(u_0) = \max_{\lambda^s \in \mathbb{R}^m} \quad \lambda^s \cdot (h^s - T^s u_0) \\
s.t. \quad (W^s)^\top \lambda^s \leq q^s
$$

admits for optimal solution $\lambda^s_{u_0}$.
Consider another control $u_0'$, we have

$$
(D_{u_0'}) \quad Q^s(u_0') = \max_{\lambda^s \in \mathbb{R}^m} \quad \lambda^s \cdot (h^s - T^s u_0') \\
s.t. \quad (W^s)^\top \lambda^s \leq q^s
$$

As $\lambda^s_{u_0}$ is admissible for $(D_{u_0})$ it is also admissible for $(D_{u_0'})$, hence

$$
Q^s(u_0') \geq \lambda^s_{u_0} \cdot (h^s - T^s u_0').
$$

# Obtaining (optimality) cuts III

To sum up we have seen that, for any admissible first stage solution, we can construct an exact cut for $Q^s$ by solving the dual of the second stage problem.

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To sum up we have seen that, for any admissible first stage solution, we can construct an exact cut for $Q^s$ by solving the dual of the second stage problem.

More precisely, let $u_0^k \geq 0$ be such that $Au_0^k = b$. Let $\lambda_k^s$ be an optimal dual solution. Then, setting

$$\alpha_k^s := -(T^s)^\top \lambda_k^s \quad \text{and} \quad \beta_k^s := (\lambda_k^s)^\top h^s$$

we have

$$\begin{cases} Q^s(u_0') \geq \alpha_k^s \cdot u_0' + \beta_k^s & \forall u_0' \geq 0, Au_0' = b \\ Q^s(u_0^k) = \alpha_k^s \cdot u_0^k + \beta_k^s \end{cases}$$

# L-shaped method (multi-cut version)

1. We have a collection of $K \times S$ cuts, such that $Q^s(u_0) \geq \alpha_k^s \cdot u_0 + \beta_k^s$.
2. Solve the master problem, with optimal primal solution $u_0^{K+1}$.

$$\begin{aligned} \min_{u_0 \geq 0} \quad & c^\top u_0 + \sum_{s=1}^{S} \pi^s \theta^s \\ s.t. \quad & A u_0 = b \\ & \theta^s \geq \alpha_k^s u_0 + \beta_k^s \qquad \forall k \in [\![1, K]\!], \quad \forall s \in [\![1, S]\!] \end{aligned}$$

3. Solve $S$ slave problems, with optimal dual solution $\lambda_{K+1}^s$

$$Q^s(u_0^{K+1}) = \min_{u_1^s \in \mathbb{R}^n} \quad q^s \cdot u_1^s$$
$$s.t. \quad W^s u_1^s = h^s - T^s u_0^{K+1}, \quad u_1^s \geq 0$$

4. construct $S$ new cuts with

$$\alpha_{K+1}^s := -(T^s)^\top \lambda_{K+1}^s, \qquad \beta_{K+1}^s := h^s \cdot \lambda_{K+1}^s$$

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3. Solve $S$ slave problems, with optimal dual solution $\lambda_{K+1}^s$

$$\begin{aligned} Q^s(u_0^{K+1}) = \max_{\lambda^s \in \mathbb{R}^m} \quad & \lambda^s \cdot \left(h^s - T^s u_0^{K+1}\right) \\ s.t. \quad & W^s \cdot \lambda^s \leq q^s \end{aligned}$$

4. construct $S$ new cuts with

$$\alpha_{K+1}^s := -(T^s)^\top \lambda_{K+1}^s, \qquad \beta_{K+1}^s := h^s \cdot \lambda_{K+1}^s$$

# L-shaped method (multi-cut version) : bounds

- At any iteration of the L-shaped method we can easily determine upper and lower bound over our problem.
- Indeed, $u_0^K$ is an admissible firt stage solution, and $Q^s(u_0^K)$ is the value of a slave problem. Thus the value of admissible solution $u_0^k$ is simply given by

$$UB = c^\top u_0^K + \sum_{s=1}^{S} \pi^s Q^s(u_0^K).$$

- Furthermore, $Q_K^s(u_0) \geq \max_{k \leq K} \alpha_k^s \cdot u_0 + \beta_k^s$, thus the value of the master problem is always a lower bound over the value of the SP problem :

$$LB = c^\top u_0^K + \sum_{s=1}^{S} \pi^s \theta_K^s.$$

# L-shaped method (single-cut version)

1. We have a collection of $K$ cuts, such that $Q(u_0) := \sum_{s \in S} Q^s(u_0) \geq \alpha_k \cdot u_0 + \beta_k$.
2. Solve the master problem, with optimal primal solution $u_0^{K+1}$.

$$
\begin{aligned}
\min_{u_0 \geq 0} \quad & c^\top u_0 + \theta \\
s.t. \quad & A u_0 = b \\
& \theta \geq \alpha_k u_0 + \beta_k \qquad \forall k \in [\![1, K]\!]
\end{aligned}
$$

3. Solve $S$ slave dual problems, with optimal dual solution $\lambda^s_{K+1}$

$$
\begin{aligned}
\max_{\lambda^s \in \mathbb{R}^m} \quad & \lambda^s \cdot \left(h^s - T^s u_0^{K+1}\right) \\
s.t. \quad & W^s \cdot \lambda^s \leq q^s
\end{aligned}
$$

4. construct new cut with

$$
\alpha_{K+1} := -\sum^{S} \pi^s \, (T^s)^\top \lambda^s, \qquad \beta_{K+1} := \sum^{S} \pi^s \, h^s \cdot \lambda^s.
$$

# Feasibility cuts

- Without the relatively complete recourse assumption we cannot guarantee that $Q(u_0) < +\infty$, however we still have that $Q$ is polyhedral, thus so is $dom(Q)$.
- Without RCR we need to add feasibility cuts in the following way:
  - If, $Q^s(u_0^k) = +\infty$, then we can find an unbounded ray of the dual problem

$$\max_{\lambda^s \in \mathbb{R}^m} \quad \lambda^s \cdot \big(h^s - T^s u_0^k\big)$$
$$s.t. \quad W^s \cdot \lambda^s \leq q^s$$

    more precisely a vector $\bar{\lambda}^k$ such that, for all $t \geq 0$ $W^s \cdot t\bar{\lambda}^k \leq q^s$.
  - Then, for $u_0$ to be admissible, we need that

$$\bar{\lambda}^k \cdot \big(h^s - T^s u_0\big) \leq 0$$

    which is a feasibility cut.

# Convergence

**Theorem**

*In the linear case, the L-Shaped algorithm terminates in finitely many steps, yielding the optimal solution.*

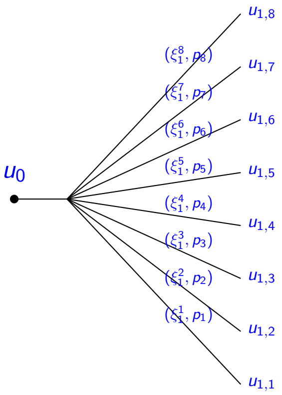
The proof is done by noting that only finitely many cuts can be added, and not being able to add a cut prove that the algorithm has converged.

# Comparison of Progressive Hedging and L-shaped

| | Progressive Hedging | L-Shaped |
|---|---|---|
| problems | convex continuous | linear, 1st stage integer |
| sol. at it. $k$ | non-admissible splitted solutions | admissible primal solution |
| Bounds | LB free, UB easy | LB and UB free |
| Convergence | asymptotic | finite |
| Complexity | fixed : S deterministic problem | increasing for master problem, fixed for slave problem |
| Implem. | easy from deterministic solver | built from scratch |

# Presentation Outline

# Where do we come from: two-stage programming




- We take decisions in two stages

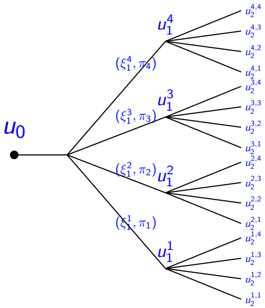
$$u_0 \rightsquigarrow \boldsymbol{\xi}_1 \rightsquigarrow \boldsymbol{u}_1 \ ,$$

with $\boldsymbol{u}_1$: recourse decision .

- On a tree, it means solving the extensive formulation:

$$\min_{u_0, u_{1,s}} c_0 u_0 + \sum_{s \in \mathbb{S}} p_s \big[ \langle c_s \ , u_{1,s} \rangle \big] \ .$$

# Extending two-stage to multistage programming




- We want to minimize $\min_{\boldsymbol{u}} \mathbb{E}\left[c(\boldsymbol{u}, \boldsymbol{\xi})\right]$
- Where we take decisions in $T$ stages

$$\boldsymbol{u}_0 \rightsquigarrow \boldsymbol{\xi}_1 \rightsquigarrow \boldsymbol{u}_1 \rightsquigarrow \cdots \rightsquigarrow \boldsymbol{\xi}_T \rightsquigarrow \boldsymbol{u}_T \ .$$

- It can be represented on a tree $\mathcal{T}$, where a node $n$ of depth $t$ represent a realisation of $(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_t)$, and to which is attached a probability $p_n$.
- Then, the extensive formulation reads

$$\min_{\{u_n\}_{n \in \mathcal{T}}} \qquad \sum_{n \in \mathcal{T}} p_n c_n(u_n)$$

# Compact and splitted extended formulation

- Consider a tree of depth $T$. A scenario $s = (n_1, \dots, n_T)$ is a sequence of node, where each element is a descendent of the previous one. A scenario $s \in S$ is uniquely defined by its last element, which is a leaf of the tree.
- Let $\pi^s$ be the probability of the leaf defining scenario $s$.
- The compact formulation of the multistage problem reads

$$\min_{\{u_n\}_{n\in\mathcal{T}}} \quad \sum_{n\in\mathcal{T}} \pi^n c_n(u_n) = \sum_{s\in S} \pi^s \sum_{n\in S} c_n(u_n)$$

- The splitted extended formulation reads

$$\begin{aligned} \min_{\{u_{s,t}\}_{s\in S, t\in [\![0,T]\!]}} \quad & \sum_{s\in S} \pi^s \sum_{t=0}^{T} c_{s,t}(u_{s,t}) \\ s.t. \quad & u_{s,t} = u_{s',t} \qquad \forall t, \forall n \in \mathcal{N}_t, \forall s, s' \ni n \end{aligned}$$

where $\mathcal{N}_t$ is the set of nodes of depth $t$

# Introducing the non-anticipativity constraint

*We do not know what holds behind the door.*

**Non-anticipativity**

At time $t$, decisions are taken sequentially, only knowing the past realizations of the perturbations.

Mathematically, this is equivalent to say that at time $t$, the decision $\boldsymbol{u}_t$ is

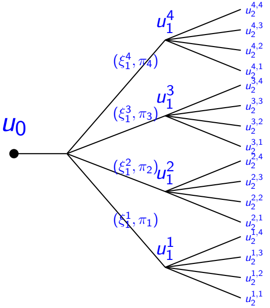
1. a function of past noises

$$\boldsymbol{u}_t = \pi_t(\boldsymbol{\xi}_0, \cdots, \boldsymbol{\xi}_t) \, ,$$

2. taken knowing the available information,

$$\sigma(\boldsymbol{u}_t) \subset \sigma(\boldsymbol{\xi}_0, \cdots, \boldsymbol{\xi}_t) \, .$$

# Multistage extensive formulation approach




Assume that $\xi_t \in \mathbb{R}^{n_\xi}$ can take $n_\xi$ values and that $U_t(x)$ can take $n_u$ values.

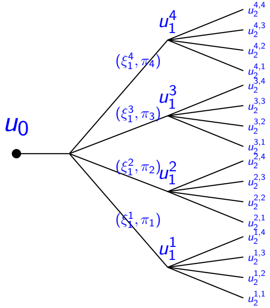
Then, considering the extensive formulation approach, we have

- $n_\xi^T$ scenarios.
- $(n_\xi^{T+1} - 1)/(n_\xi - 1)$ nodes in the tree.
- Number of variables in the optimization problem is roughly $n_u \times (n_\xi^{T+1} - 1)/(n_\xi - 1) \approx n_u n_\xi^T$.

The complexity grows exponentially with the number of stage. :-(

A way to overcome this issue is to compress information!

# Multistage extensive formulation approach

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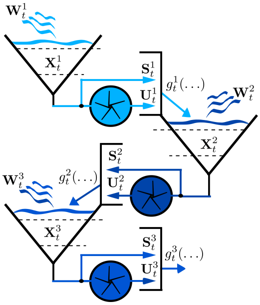
# Illustrating extensive formulation with the damsvalley example




- 5 interconnected dams
- 5 controls per timesteps
- 52 timesteps (one per week, over one year)
- $n_\xi = 10$ noises for each timestep

We obtain $10^{52}$ scenarios, and $\approx 5.10^{52}$ constraints in the extensive formulation ... Estimated storage capacity of the Internet: $10^{24}$ bytes.

## 2-stage approach

The 2-stage approach consists in approximating the multistage program by a two-stage programm :

- relax all non-anticipativity constraints except the ones on $u_0$, this turn the tree into a scenario fan (same number of scenario),
- it means that all decision $(u_1, \dots, u_{T-1})$ are anticipative (not $u_0$).
- reduce the number of scenarios by sampling, and solve the SAA approximation of the 2-stage relaxation.

Denote $v^\sharp$ the value of the multistage problem, $v^{2SA}$ the value of the 2-stage relaxation, and $v_m^{2SA}$ the (random) value of the SAA of the 2-stage relaxation. Then we have

$$v^{2SA} \leq v^\sharp$$

$$v_m^{2SA} \to v^{2SA}$$

$$\mathbb{E}\left[v_m^{2SA}\right] \leq v^{2SA}$$