Stochastic Optimization - 2 hours

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This exam has two independent parts. Course documents authorized. No communication with other people.

1 Energy production (8 points)

Consider an energy producer which has a random demand d to meet or excess. He can produce energy q (in MWh) for a cost q^2 through a thermal engine that has to be set in advance. If the production is not enough he can buy energy b, on the spot (i.e. even at the last minute) the missing energy for $p_1 b$, with $0 \leq b \leq 2.5MWh$.

Demand is assumed to be $1MWh$ with probability 0.5, $2MWh$ with probability 0.25 and $3MWh$ with probability 0.25.

1. (1 point) Justify that this problem is a two-stage stochastic program by specifying first and second stage variable and write the extensive formulation of this problem.

Solution: q is the first stage control, b the second stage control. (0.5) The extensive form reads: (0.5)

$$\begin{aligned} \min_{q,b_1,b_2,b_3 \geq 0} \quad & q^2 + 0.5p_1b_1 + 0.25p_1b_2 + 0.25p_1b_3 \\ \text{s.t.} \quad & b_s \leq 2.5 & \forall s \in [3] \\ & q + b_1 \geq 1 \\ & q + b_2 \geq 2 \\ & q + b_3 \geq 3 \end{aligned}$$

2. (1.5 points) Write a time-decomposition of the problem that is in a relatively complete recourse setting.

Solution: We have (1)

$$\begin{aligned} \min_{q} \quad & q^2 + 0.5\hat{V}(q, 10) + 0.25\hat{V}(q, 20) + 0.25\hat{V}(q, 30) \\ \text{s.t.} \quad & q \geq 0.5 \end{aligned}$$

with

$$\hat{V}(q, d) = \min \quad \{p_1b_3 \quad | \quad 0 \leq b \leq 25, \quad q + b \geq d\}$$

The constraint $q \geq 0.5$ ensure relatively complete recourse. (0.5)

3. (1.5 points) Assuming that $p_1 = 4$, compute the anticipative lower bound.

Solution: For a given scenario the optimal cost is given by $\min_{q \geq 0.5} q^2 + 4(d-q)^+$. For any d, the optimal q is either d(0.25) (with cost d^2) or such that $2q - 4 = 0$ that is $q = 2$(0.25), and cost $q^2 + 4(d - 2)$.

Consequently the optimal first stage for each scenario is $q_1 = 1$, $q_2 = q_3 = 2$, (0.5) with cost 1, 4 and 8, leading to an anticipative lower bound of $0.5 + 1 + 2 = 3.5$.(0.5)

4. (2 points) Can we apply the Progressive Hedging algorithm to this problem ? If so give the master problem and slave problems at iteration k. If not Justify.

Solution: The problem is convex and risk neutral, hence we can apply the Progressive Hedging algorithm.(0.5)

Each slave problems read (1)

$$\begin{aligned} \min_{q_i,b_i} \quad & q_i^2 + 4b_i + \lambda_i^k q_i + \rho(q_i - \bar{q}^{k-1})^2 \\ \text{s.t.} \quad & q_i + b_i \geq d_i \\ & q_i \geq 0.5, \quad b_i \geq 0 \end{aligned}$$

and the master problem read (0.5)

$$\bar{q}^k = 0.5q_1^k + 0.25q_2^k + 0.25q_3^k \qquad \text{and} \qquad \lambda^{k+1} = \lambda^k + \rho(q_i - \bar{q}^k)$$

5. (2 points) Slightly adapt the L-Shaped method to this problem. Gives the master problem and slave problems (in primal form) at iteration k (multicut version).

Solution: The first stage problem is quadratic thus we are not in the classical L-shaped setting, however the algorithm still work.

The master problem read (1)

$$\begin{aligned} \min_{q,\theta_1,\theta_2,\theta_3} \quad & q^2 + 0.5\theta_1 + 0.25\theta_2 + 0.25\theta_3 \\ \text{s.t.} \quad & \theta_i \geq (\alpha_i^\kappa)^\top q + \beta_i^\kappa & \kappa \leq k \\ & q \geq 0.5 \end{aligned}$$

with slave problems (1)

$$\begin{aligned} \min_{\tilde{q},b_i} \quad & 4b_i \\ \text{s.t.} \quad & b_i \geq d_i - \tilde{q} \\ & \tilde{q} = q & [\alpha_i] \end{aligned}$$

2 A stochastic SOCP (13 points)

A second order cone program (SOCP) is an optimization problem of the form

$$\begin{aligned}
(SOCP) \qquad & \min_{z\in\mathbb{R}^n} && q^\top z \\
& s.t. && \|A_i z + b_i\|_2 \le c_i^\top z + d_i && \forall i = 1,..,k \\
& && A_0 z = b_0 \\
& && A_1 z \le b_1
\end{aligned}$$

In particular every linear program are SOCP, and you can add some constraints with $\|\|_2$ norm.

We now consider the following stochastic optimization problem (all equalities hold almost surely)

$$\begin{aligned}
(P) \qquad & \min && c^\top x_0 + \mathbb{E}\left[\sum_{t=0}^{T-1} q_t \|\boldsymbol{u}_t\|_2\right] \\
& s.t. && \boldsymbol{x}_{t+1} = \boldsymbol{x}_t + H\boldsymbol{u}_t && \forall t = 0, ..., T-1 \\
& && \|\boldsymbol{x}_T\| \le R && \mathbb{P} - a.s. \\
& && A\boldsymbol{x}_t + B\boldsymbol{u}_t \le \boldsymbol{b}_t \\
& && \sigma(\boldsymbol{x}_t, \boldsymbol{u}_t) \subset \sigma(\boldsymbol{b}_0, \dots, \boldsymbol{b}_t)
\end{aligned}$$

Where $\boldsymbol{x}_t \in \mathbb{R}^3$ and $\boldsymbol{u}_t \in \mathbb{R}^{10}$. We assume that the noise $(\boldsymbol{b}_t)_t$ is a sequence of independent random variables, such that, for each t, $\boldsymbol{b}_t$ has a finite support of size 4.

1. We start with some generic comments on (P).
   (a) (0.5 points) Show that an SOCP is a convex problem

   Solution: $\|\cdot\|$ is a convex function, the composition of convex and affine is convex, thus the constraint set is convex. The objective function being linear, an SOCP is a convex problem.

   (b) (0.5 points) What is the state, the control, the noise affecting (P) ?

   Solution: state $\boldsymbol{x}$, control $\boldsymbol{u}$ and noise $\boldsymbol{b}$.

2. In this question we assume that $T = 2$.
   (a) (3 points) Write the extensive formulation of problem (P) as a finite dimensional SOCP. Give the number of optimization variables and the number of conic constraints with their size (i.e the dimension of the vector inside the norm).

   Solution: Assume that $\boldsymbol{b}$ can take value b_i with probability π_i for $i \in [4]$. Then the extensive formulation (compact form) of (P) reads (2)

$$\begin{aligned}
\min_{x,u,\theta} \quad & c^\top x^0 + \sum_{i=1}^4 \pi_i q_0 \theta_i^1 + \sum_{i,j=1}^4 \pi_i\pi_j q_2 \theta_{i,j}^2 \\
\text{s.t.} \quad & \|u_i^0\| \le \theta_i^1 && \forall i \in [4] \\
& \|u_{i,j}^1\| \le \theta_{i,j}^2 && \forall i,j \in [4]^2 \\
& x_i^1 = x^0 + Hu_i^0, && \forall i \in [4] \\
& x_{i,j}^2 = x_i^1 + Hu_{i,j}^1 && \forall i,j \in [4]^2 \\
& \|x_{i,j}^2\| \le R && \forall i,j \in [4]^2 \\
& Ax^0 + Bu_i^0 \le b_i, && \forall i \in [4] \\
& Ax_i^1 + Bu_{i,j}^1 \le b_j && \forall i,j \in [4]^2
\end{aligned}$$

There are 20 θ variables of size 1, $4+16=20$ variables u of size 3 and 21 variables x of size 3, hence 283 variables. (0.5) Further, there are 16 conic constraints of size 3 and 20 conic constraints of size 10. (0.5)

(b) (2 points) Write the two-stage approximation $(2SA)$ of (P) as a SOCP. Is it simpler to solve than the full problem ?

Solution: The two stage approximation reads (1.5)

$$
\begin{aligned}
\min_{x,u,\theta} \quad & c^\top x^0 + \sum_{i,j=1}^{4} \pi_i \pi_j (q_0 \theta^1_{i,j} + q_2 \theta^2_{i,j}) \\
\text{s.t.} \quad & \|u^0_{i,j}\|_2 \leq \theta^1_{i,j} && \forall i,j \in [4]^2 \\
& \|u^1_{i,j}\| \leq \theta^2_{i,j} && \forall i,j \in [4]^2 \\
& x^1_{i,j} = x^0 + H u^0_{i,j}, && \forall i,j \in [4]^2 \\
& x^2_{i,j} = x^1_{i,j} + H u^1_{i,j} && \forall i,j \in [4]^2 \\
& \|x^2_{i,j}\|_2 \leq R && \forall i,j \in [4]^2 \\
& A x^0 + B u^0_{i,j} \leq b_{i,j}, && \forall i,j \in [4]^2 \\
& A x^1_{i,j} + B u^1_{i,j} \leq b_j && \forall i,j \in [4]^2
\end{aligned}
$$

It is more difficult to solve than (P) as it requires more variable and constraints. (0.5)

(c) (1 point) Would solving a SAA version of $(2SA)$ be faster ? Justify.

Solution: No it would not as there are only 16 scenarios here, and SAA would require more (say at least 100) to have some precision.

3. We now consider the case of large T.

(a) (2 points) Give a Bellman operator $\mathcal{B}$ associated to problem (P), and the dynamic programming equation. Do not forget the initialization of the recursion with the final cost. (You might write a special problem for $t=0$).

Solution: We have (0.5)

$$
\begin{cases}
V_T(\cdot) = \mathbb{I}_{\|\cdot\| \leq R} \\
V_t = \mathcal{B}_t(V_{t+1}) \qquad \forall t \in [\![1, T-1]\!]
\end{cases}
$$

with (0.5)

$$
\mathcal{B}_t(R) = \mathbb{E}\big[\hat{\mathcal{B}}_t(R)(x, \boldsymbol{b})\big].
$$

where (0.5)

$$
\begin{aligned}
\hat{\mathcal{B}}_t(R)(x,b) = \min_{u,\theta} \quad & q_t \theta + R(y) \\
\text{s.t.} \quad & y = x + Hu \\
& \|u\| \leq \theta \\
& Ax + Bu \leq b.
\end{aligned}
$$

Finally, the first stage problem reads (0.5)

$$
\min_{x^0} \quad c^\top x^0 + V_0(x^0).
$$

(b) (0.5 points) Is computing $\mathcal{B}_t(V_{t+1})(x_t)$ equivalent to solving a SOCP ?

Solution: No as we cannot guarantee that V_{t+1} is SOCP representable.

(c) (2.5 points) Assuming that you have an efficient SOCP solver, give a pseudo-code for solving (P) by Dynamic Programming through a discretization / interpolation approach. Is it reasonable ?

Solution: V_t are convex, thus a convex interpolation is possible. An algorithm would be (2)

Data: Discretization points X^D
for $t = T-1 \to 1$ **do**
  for $x \in X^D$ **do**
    $V_x^t = 0$;
    for $i \in [4]$ **do**

$$\begin{aligned} \hat{v} = \min_{u,\theta,\sigma} \quad & q_t\theta + \sum_{z \in X^D} \sigma_z V_z^{t+1} \\ \text{s.t.} \quad & y = \sum_{z \in X^D} \sigma_z z \\ & y = x + Hu \\ & Ax + Bu \leq b_i \\ & \sigma \geq 0, \qquad \sum_{x \in X^D} \sigma_x = 1 \end{aligned}$$

      $V_x^t \leftarrow V_x^t + \pi_i \hat{v}$;
    end
  end
end

Algorithm 1: A discretized DP algorithm

This approach is reasonable as the state is only in dimension 3. (0.5)

(d) (1 point) Can you use SDDP to solve (P) ? Justify.

Solution: The problem is convex thus SDDP could be used if the SOCP solver return optimal multiplier (which it does). (0.5) However SDDP requires the state and control set to be bounded, which is not obvious here, and relatively complete recourse. Relatively complete recourse can not be guaranteed here (0.5).