Stochastic Optimization - 2 hours

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This exam has two parts. Course documents authorized. No communication with other peoples.

1 Fast and slow product ordering (7 points)

Consider a company that can order a product whose demand is uncertain by boat in advance (unlimited quantity) at price 1 per unit or by plane at price 4 per unit once the demand is known (maximum of 15 units). Demand has to be met. Unsold product is lost. We aim at minimizing the expected ordering cost.

Demand is assumed to be 10 with probability 0.4, 20 with probability 0.4 and 30 with probability 0.2.

1. (1 point) Justify that this problem is a two-stage stochastic programm by specifying first and second stage variable and write the extensive formulation of this problem.
2. (1 point) Are we in a complete recourse setting? A relatively complete recourse setting? If not give an explicit first stage constraint yielding relatively complete recourse.
3. (1 point) Give the anticipative lower bound.
4. (2 points) Can we apply the L-Shaped method to this problem? If so gives the master problem and slave problems at iteration k (multicut version).
5. (2 points) Can we apply the Progressive Hedging algorithm to this problem? If so give the master problem and slave problems at iteration k.

2 A quadratically constrained problem (14 points)

A quadratically constrained quadratic program (QCQP) is an optimization problem of the form

$$
\begin{aligned}
(QCQP) \qquad & \min_{x\in\mathbb{R}^n} && \frac{1}{2}x^\top P_0 x + q_0^\top x \\
& s.t. && \frac{1}{2}x^\top P_i x + q_i^\top x + r_i \leq 0 && \forall i = 1,..,k \\
& && Ax = b
\end{aligned}
$$

We consider the following stochastic optimization problem (all equality hold almost surely)

$$
\begin{aligned}
(P) \qquad & \min && \mathbb{E}\Big[\sum_{t=0}^{T-1} \frac{1}{2}\boldsymbol{u}_t^\top \boldsymbol{P}_t \boldsymbol{u}_t + \boldsymbol{q}_t \boldsymbol{u}_t\Big] \\
& s.t. && \boldsymbol{x}_{t+1} = \boldsymbol{x}_t + R\boldsymbol{u}_t && \forall t = 0, ..., T-1 \\
& && \boldsymbol{x}_T = x_f \\
& && \boldsymbol{x}_t^\top A \boldsymbol{x}_t + b^\top \boldsymbol{x}_t + c \leq 0 \\
& && \sigma(\boldsymbol{x}_t, \boldsymbol{u}_t) \subset \sigma(\boldsymbol{P}_0, \boldsymbol{q}_0, \dots, \boldsymbol{P}_t, \boldsymbol{q}_t)
\end{aligned}
$$

Where $\boldsymbol{x}_t \in \mathbb{R}^3$ and $\boldsymbol{u}_t \in \mathbb{R}^{10}$, and all square matrices are semi-definite positive. We assume that $\boldsymbol{x}_0$ is fixed at 0. We assume that the noise $(\boldsymbol{P}_t, \boldsymbol{q}_t)_t$ is a sequence of independant random variables, such that, for each t, $(\boldsymbol{P}_t, \boldsymbol{q}_t)_t$ has a finite support of size 4.

1. We start with some generic comments on (P).
   (a) (1 point) Show that if all P_i are semi definite positive then $(QCQP)$ is convex.
   (b) (1 point) What is the state, the control, the noise affecting (P)? What is the information structure of (P)?
   (c) (1 point) Give a simple condition ensuring that we are in a relatively complete recourse framework.
2. In this question we assume that $T = 3$.
   (a) (2 points) Write problem (P) as a finite dimensional convex QCQP (objective function can be given as a sum of quadratic costs). Give the number of optimization variables and the number of quadratic constraints with their size.
   (b) (2 points) Write the two-stage approximation $(2SA)$ of (P) as a QCQP. Is it simpler to solve than the full problem?
   (c) (1 point) Would solving a SAA version of $(2SA)$ be faster? Justify.
3. We now consider the case of large T.
   (a) (2 points) Give a Bellman operator $\mathcal{B}$ associated to problem (P), and the dynamic programming equation (dont forget the initialization of the recursion with the final cost).
   (b) (1 point) Is computing $\mathcal{B}_t(V_{t+1})(x_t)$ equivalent to solving a QCQP?
   (c) (2 points) Assuming that you have an efficient QCQP solver, give a pseudo-code for solving (P) by Dynamic Programming through a discretization / interpolation approach. Is it reasonable?
   (d) (1 point) Can you use SDDP to solve (P)? Justify.