# Stochastic Optimization - 2 hours

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16/01/2018

Name: ____________________

By "type of problem", we mean "Linear Program" denoted (LP), "Mixed Integer Linear Program" denoted (MILP), "Quadratic Program" denoted (QP) (linear constraint, convex quadratic objective, continuous variables), "Mixed Integer Quadratic Program" denoted (MIQP)... By size of a problem we mean the number of integer and linear variables.

## 1 A cutting plane algorithm for convex two stage programm (12 points)

We consider the following two stage stochastic programm

$$
\begin{array}{lll}
(SP) & \min_{q,\boldsymbol{u}} & J(q) + \mathbb{E}\big[c^T \boldsymbol{u}\big] \\
 & s.t. & q \in Q \\
 & & A\boldsymbol{u} \leq \boldsymbol{b} + Bq \\
 & & \sigma(\boldsymbol{u}) \preceq \sigma(\boldsymbol{b})
\end{array}
$$

where $J : \mathbb{R}^{n_u} \to \mathbb{R}$ is a convex fonction, $c \in \mathbb{R}^{n_u}$ a deterministic cost vector, $Q \subset \mathbb{R}^{n_q}$ is a polytope (compact polyhedron), $A$ and $B$ deterministic matrices, and $\boldsymbol{b}$ a random vector with known discrete probability law : $\mathbb{P}(\boldsymbol{b} = b_i) = \pi_i$ for $i \in [\![1, n]\!]$.

1. (1 point) Justify that this problem is a two-stage stochastic programm by giving the first and second stage decision variable and the noise.

2. (2 points) Define $V(q) = \mathbb{E}\big[\hat{V}(q, \boldsymbol{b})\big]$ where

$$
\begin{array}{lll}
\hat{V}(q, b) = & \min_{u \in \mathbb{R}^{n_u}} & c^T u \\
 & s.t. & Au \leq \boldsymbol{b} + Bq
\end{array}
$$

With this definition propose a simple formulation of $(SP)$. Give an explicit (i.e. without domain of some function) relatively complete recourse condition

3. (3 points) For $i \in [\![1, n]\!]$, and $q \in Q$, write a linear program with value $\hat{V}(q, b_i)$ and show how to obain $\lambda_i \in \partial_q \hat{V}(q, b_i)$ from an optimal solution.

4. (2 points) For $q \in Q$, explain how to efficiently obtain $\alpha \in \mathbb{R}^{n_q}$ and $\beta \in \mathbb{R}$ such that, for all $q' \in Q$ , $\alpha^T q' + \beta \leq V(q')$ and $\alpha^T q + \beta = V(q)$. Precise the number, type and size of problems solved.

5. (3 points) Assuming that you are only allowed an LP solver, propose an algorithm solving $(SP)$ and guarantee its convergence (using the results from the course). Precise the number, type and size of problems solved at each iteration.

6. (2 points) Give the pseudo-code of a multicut version of your algorithm.

# 2 A multi-stock problem (17 points)

Consider a chemical industry that use 10 different raw materials, to produce 5 different finished products. Denote by $a_{r,f}$ the quantity (in tons) of raw material $r$ required to produce one ton of finished product $f$. At any point in time, the company cannot have more than 7 tons of each raw material. Each month, at the beginning of the month, the company discover the vector of cost $\boldsymbol{c}_t$ at which it can order any mix of raw material for a maximum of 20 tons in total, and the vector of prices $\boldsymbol{p}_t$ at which finished product will be sold (every finished product is sold). Material ordered at the beginning of the month arrive at the end of the month. The prices $\boldsymbol{p}_t$ and $\boldsymbol{c}_t$ are highly correlated, but independent in time and the law of $\boldsymbol{\xi}_t = (\boldsymbol{p}_t, \boldsymbol{c}_t)$ is finitely supported with a support $\Xi$ of size 100 (we denote $\pi_\xi = \mathbb{P}(\boldsymbol{\xi} = \xi)$ for all $\xi \in \Xi$).

The company want to optimize its expected gain over one year, starting with 5 tons of each raw material. Stock at the end of the year are considered lost. We will denote by $b_t \in \mathbb{R}^{10}$ the quantity of raw material bought at month $t$, by $v_t \in \mathbb{R}^5$ the quantity of finished product sold during month $t$, and $x_t$ the quantity of raw material available at the beginning of month $t$.

1. (0.5 points) Give a matrix $A$ such that we have $\boldsymbol{x}_{t+1} = \boldsymbol{x}_t + \boldsymbol{b}_t - A\boldsymbol{v}_t$.

2. (2.5 points) Write the problem as a multistage stochastic optimization problem. Precise the information structure.

3. (2 points) Justify that we can apply Dynamic Programming to this problem. Define adequate Bellman operators $\hat{\mathcal{B}}_t$ and $\mathcal{B}_t$ and give the associated Bellman Equation (beware of constraints on the stock).

4. Dynamic Programming for a discretized version. In this question we consider that the state $x$ is constrained to be a vector of integer.

   (a) (2 points) We call $\Psi(x_t, x_{t+1}, \xi)$ the minimal cost of going from state $x_t$ at the beginning of month $t$ to state $x_{t+1}$ knowing that the prices are given by $\xi = (p_t, c_t)$. Write a mathematical program computing $\Psi(x_t, x_{t+1}, \xi)$, and precise its type.

(b) (3 points) Using function $\Psi$, propose a Dynamic Programming algorithm on the discretized problem. How many time is the function $\Psi$ called? Is it a reasonable approach?

5. Stochastic Dual Dynamic Programming

(a) (1 point) Justify that SDDP is adapted to the continuous problem (in particular the recourse assumption).

(b) (2 points) Consider the forward pass $k$ of the SDDP algorithm at time $t$, give the LP problem solved ("one-stage-one-alea" problem).

(c) (1 point) How many times is this (type of) problem solved during iteration $k$ of SDDP?

(d) (1 point) What is the "one-stage-one-alea" LP problem solved for a multicut version of the problem?