Wardrop Equilibrium

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Contents

Convex set

- A set $C \subset \mathbb{R}^n$ is *convex* iff

$$\forall x, y \in C, \quad \forall t \in [0,1], \qquad tx + (1-t)y \in C.$$

- Intersection of convex sets is convex.
- A closed convex set C is equal to the intersection of all half-spaces containing it.

Convex function

- The *epigraph* of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is

$$\text{epi}(f) := \Big\{(x,t) \in \mathbb{R}^n \times \mathbb{R} \quad | \quad t \geq f(x)\Big\}.$$

- The *domain* of a function f is

$$\mathrm{dom}(f) := \Big\{x \in \mathbb{R}^n \quad | \quad f(x) < +\infty\Big\}$$

- The function f is said to be *convex* iff its epigraph is convex, in other words iff

$$\forall t \in [0,1], \qquad f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

- The function f is said to be *strictly convex* iff

$$\forall t \in (0,1), \qquad f(tx + (1-t)y) < tf(x) + (1-t)f(y).$$

Convexity and differentiable

We assume sufficient regularity for the written object to exist.

- If $f : \mathbb{R} \to \mathbb{R}$.
  - f is convex iff f' non-decreasing.
  - If f' strictly increasing then f is strictly convex.
  -
  - f is above its tangeants : $f(y) \geq f(x) + f'(x)(y-x)$.
  - f is convex iff $f'' \geq 0$.
  - If $f'' > 0$ then f is strictly convex.
- If $f : \mathbb{R}^n \to \mathbb{R}$
  - f is convex iff ∇f non-decreasing (i.e. $(\nabla f(y) - \nabla f(x)) \cdot (y-x) \geq 0$).
  - f is above its tangeants : $f(y) \geq f(x) + \nabla f(x) \dot (y-x)$.
  - f is convex iff $\nabla^2 f(x) \succeq 0$ for all x.
  - If $\nabla^2 f(x) \succ 0$ for all x then f is strictly convex.

Video explanation

https://www.youtube.com/watch?v=qF0aDJfEa4Y

Convex differentiable optimization problem

Consider the following optimization problem.

$$
\begin{aligned}
\min_{x\in\mathbb{R}^n} \quad & f(x) && (P) \\
\text{s.t.} \quad & g_i(x) = 0 && \forall i \in [n_E] \\
& h_j(x) \leq 0 && \forall j \in [n_I]
\end{aligned}
$$

with

$$X := \{x \in \mathbb{R}^n \mid \forall i \in [n_E], \quad g_i(x) = 0, \quad \forall j \in [n_I], \quad h_j(x) \leq 0\}.$$

- (P) is a *convex optimization problem* if f and X are convex.
- (P) is a *convex differentiable optimization problem* if f, and h_j (for $j \in [n_I]$) are convex differentiable and g_i (for $i \in [n_E]$) are affine

Convex differentiable optimization problem

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KKT conditions

Theorem (KKT)

Let $x^\sharp$ be an optimal solution to a differentiable optimization problem (P). If the constraints are qualified at $x^\sharp$ then there exists optimal multipliers $\lambda^\sharp \in \mathbb{R}^{n_E}$ and $\mu^\sharp \in \mathbb{R}^{n_I}$ satisfying

$$\begin{cases} \nabla f(x^\sharp) + \sum_{i=1}^{n} \lambda_i^\sharp \nabla g_i(x^\sharp) + \sum_{j=1}^{n_I} \mu_i^\sharp \nabla h_j(x^\sharp) = 0 & \textit{first order condition} \\ g(x^\sharp) = 0 & \textit{primal admissibility} \\ h(x^\sharp) \leq 0 & \\ \mu \geq 0 & \textit{dual admissibility} \\ \mu_i h_i(x^\sharp) = 0, \quad \forall i \in [n_I] & \textit{complementarity} \end{cases}$$

The three last conditions are sometimes compactly written

$$0 \geq h(x^\sharp) \perp \mu \geq 0.$$

Video explanation (at a later time)

Intro to constrained optimization
https://www.youtube.com/watch?v=vwUV2IDLP8Q
Explaining tangeancy of multipliers
https://www.youtube.com/watch?v=yuqB-d5MjZA
Marginal interpretation of multipliers
https://www.youtube.com/watch?v=m-G3K2GPmEQ

Slater condition

A convex optimization problem (P) satisfies the *Slater* condition if there exists a strictly admissible $x_0 \in \mathbb{R}^n$ that is

$$\forall i \in [n_E], \quad g_i(x_0) = 0, \quad \forall j \in [n_I], \quad h_j(x_0) < 0.$$

If the Slater condition is satisfied, then the constraints are qualified at any $x \in X$.

Another optimality condition (convex case)

Theorem

If (P) *is a convex differentiable optimization problem, then* $x^\sharp \in X$ *is an optimal solution iff*

$$\forall y \in X, \quad \nabla f(x) \cdot (y - x) \geq 0.$$

Contents

The set-up

- $G = (V, E)$ is a directed graph
- x_e for $e \in E$ represent the flux (number of people per hour) taking edge e
- $\ell_e : \mathbb{R} \to \mathbb{R}^+$ the cost incurred by a given user to take edge e
- We consider K origin-destination vertex pair $\{o^k, d^k\}_{k \in [K]}$, such that there exists at least one path from o^k to d^k.
- r_k is the rate of people going from o^k to d^k
- $\mathcal{P}_k$ the set of all simple (i.e. without cycle) path form o^k to d^k
- We denote f_p the flux of people taking path $p \in \mathcal{P}_k$

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Some physical relations

People going from o^k to d^k have to choose a path

$$r^k = \sum_{p \in \mathcal{P}^k} f_p.$$

People going through an edge are on a simple path taking this edge

$$x_e = \sum_{p \ni e} f_p.$$

The flux are non-negative

$$\forall p \in \mathcal{P}, \quad f_p \geq 0, \qquad \text{and} \qquad \forall e \in E, \quad x_e \geq 0$$

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System optimum problem

The system optimum consists in minimizing the sum of all costs over the admissible flux $x = (x_e)_{e \in E}$

- Given x, the cost of taking edge e for one person is $\ell_e(x_e)$.
- The cost for the system for edge e is thus $x_e \ell_e(x_e)$.
- Thus minimizing the system costs consists in solving

$$
\begin{aligned}
\min_{x,f} \quad & \sum_{e \in E} x_e \ell_e(x_e) & \quad (SO) \\
s.t. \quad & r_k = \sum_{p \in \mathcal{P}_k} f_p & k \in [K] \\
& x_e = \sum_{p \ni e} f_p & e \in E \\
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Path intensity formulation

- We can reformulate the (SO) problem only using path-intensity $f = (f_p)_{p \in \mathcal{P}}$.
- Define $x_e(f) := \sum_{p \ni e} f_p$, and $x = (x_e)_{e \in E}$.
- Define the loss along a path $\ell_p(f) = \sum_{e \in p} \ell_e\big(\underbrace{\sum_{p' \ni e} f_{p'}}_{x_e(f)}\big)$
- The total cost is thus

$$C(f) = \sum_{p \in \mathcal{P}} f_p \ell_p(f) = \sum_{e \in E} x_e \ell_e(x_e(f)) = C(x(f)).$$

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Path intensity problem

$$\begin{aligned}
\min_{f} \quad & \sum_{p \in \mathcal{P}} f_p \ell_p(f) && (SO) \\
s.t. \quad & r_k = \sum_{p \in \mathcal{P}_k} f_p && k \in [K] \\
& f_p \geq 0 && p \in \mathcal{P}
\end{aligned}$$

Equilibrium definition

John Wardrop defined a traffic equilibrium as follows. "Under equilibrium conditions traffic arranges itself in congested networks such that all used routes between an O-D pair have equal and minimum costs, while all unused routes have greater or equal costs."

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A mathematical definition reads as follows.

Definition

A user flow f is a User Equilibrium if

$$\forall k \in [K], \quad \forall (p, p') \in \mathcal{P}_k^2, \qquad f_p > 0 \quad \Longrightarrow \quad \ell_p(f) \leq \ell_{p'}(f).$$

A new cost function

We are going to show that a user-equilibrium f is defined as a vector satisfying the KKT conditions of a certain optimization problem.

Let define a new edge-loss function by

$$L_e(x_e) := \int_0^{x_e} \ell_e(u)du.$$

The Wardrop potential is defined (for edge intensity) as

$$W(f) = W(x(f)) = \sum_{e \in E} L_e(x_e(f)).$$

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User optimum problem

Theorem

A flow f is a user equilibrium if and only if it satisfies the first order KKT conditions of the following optimization problem

$$
\begin{aligned}
\min_{x,f} \quad & W(x) & \\
s.t. \quad & r_k = \sum_{p \in \mathcal{P}_k} f_p & k \in [K] \\
& x_e = \sum_{p \ni e} f_p & e \in E \\
& f_p \geq 0 & p \in \mathcal{P}
\end{aligned}
$$

Proof

In path intensity formulation

$$\begin{aligned} \min_f \quad & \sum_{e\in E} L_e\Big(\sum_{p\ni e} f_p\Big) & \\ s.t. \quad & r_k = \sum_{p\in\mathcal{P}_k} f_p & k\in[K] \\ & f_p \geq 0 & p\in\mathcal{P} \end{aligned}$$

with Lagrangian

$$L(f,\lambda,\mu) := W(f) + \sum_{k=1}^K \lambda_k\Big(r_k - \sum_{p\in\mathcal{P}_k} f_p\Big) + \sum_{p\in\mathcal{P}} \mu_p f_p.$$

Proof

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Proof II

Now note that we have

$$\begin{aligned}\frac{\partial W}{\partial f_p}(f) &= \frac{\partial}{\partial f_p}\Big(\sum_{e\in E} L_e(\sum_{p'\ni e} f_{p'})\Big)\\ &= \sum_{e\in p}\frac{\partial}{\partial x_e}L_e(x_e(f))\\ &= \sum_{e\in p}\ell_e(x_e(f)) = \ell_p(f),\end{aligned}$$

Recall that $L_e(x_e) := \int_0^{x_e} \ell_e(u)du$.

Proof III

The constraints of (UE) are qualified. First-order KKT conditions reads

$$\begin{cases} \dfrac{\partial L(f,\lambda,\mu)}{\partial f_p} = \ell_p(f) - \lambda_k + \mu_p = 0 & \forall p \in \mathcal{P}_k, \forall k \in [\![1,K]\!] \\ \dfrac{\partial L(f,\lambda,\mu)}{\partial \lambda_k} = r_k - \sum_{p\in\mathcal{P}_k} f_p = 0 & \forall k \in [\![1,K]\!] \\ \mu_p = 0 \text{ or } f_p = 0 & \forall p \in \mathcal{P} \\ \mu_p \leq 0, f_p \geq 0 & \forall p \in \mathcal{P} \end{cases}$$

f satisfies the KKT conditions iff for all origin-destination pair $k \in [K]$, and all path $p \in \mathcal{P}_k$ we have

$$\begin{cases} \ell_p(f) = \lambda_k & \text{if } f_p > 0 \\ \ell_p(f) \geq \lambda_k & \text{if } f_p = 0 \end{cases}$$

In other words, if the path $p \in \mathcal{P}_k$ is used, then its cost is λ_k, and all other path $p' \in \mathcal{P}_i$ have a greater or equal cost, which is the definition of a User Equilibrium.

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Convex case : equivalence

If the loss functions (in edge-intensity) are non-decreasing then the Wardrop potential W is convex.

Theorem

Assume that the loss function ℓ_e are non-decreasing for all $e \in E$. Then there exists at least one user equilibrium, and a flow f is a user equilibrium if and only if it solves (UE)

Proof : the cost is convex as composition of convex and affine functions, thus KKT is a necessary and sufficient condition for optimality.

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Convex case : characterization

define the system cost of a flow f for a given flow f', as

$$C^{f'}(f) := \sum_{e \in E} x_e(f)\ell_e\big(x_e(f')\big).$$

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Theorem

Assume that the cost functions ℓ_e are continuous and non-decreasing. Then, f^{UE} is a user equilibrium iff

$$\forall f \in F^{ad}, \qquad C^{f^{UE}}(f^{UE}) \leq C^{f^{UE}}(f),$$

where F^{ad} is the set of admissible flows.

Proof

By convexity (f^{UE}) is an optimal solution to (UE) iff

$$\nabla W(f^{UE}) \cdot (f - f^{UE}) \geq 0, \qquad \forall f \in F^{ad}$$

which is equivalent to

$$\sum_{p \in \mathcal{P}} \underbrace{\frac{\partial W}{\partial f_p}(f^{UE})}_{\ell_p(f^{UE})} f_p \quad \geq \quad \sum_{p \in \mathcal{P}} \underbrace{\frac{\partial W}{\partial f_p}(f^{UE})}_{\ell_p(f^{UE})} f_p^{UE}, \qquad \forall f \in F^{ad}$$

which can be written

$$C^{f^{UE}}(f^{UE}) \leq C^{f^{UE}}(f), \qquad \forall f \in F^{ad}.$$

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Contents

Definition

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Consider increasing loss functions ℓ_e. Let f^{UE} be a user equilibrium, and f^{SO} be a system optimum. Then the price of anarchy of our network is given by

$$PoA := \frac{C(f^{UE})}{C(f^{SO})} \geq 1.$$

Theorem

Let ℓ_e be the affine function $x_e \mapsto b_e x_e + c_e$, with $b_e, c_e \geq 0$. Then the price of anarchy is lower than $4/3$, and the bound is tight.

Proof

Let f be a feasible flow, and f^{UE} be the user equilibrium. For ease of notation we fix $x^{UE} = x(f^{UE})$, and $x = x(f)$.

By Theorem we have

$$
\begin{aligned}
C(f^{UE}) &\leq C^{f^{UE}}(f) \\
&= \sum_{e\in E} (b_e x_e^{UE} + c_e) x_e \\
&\leq \sum_{e\in E} \Big[(b_e x_e + c_e) x_e + \frac{1}{4} b_e (x_e^{UE})^2 \Big] \qquad \text{as } (x_e - x_e^{UE}/2)^2 \geq 0 \\
&\leq C(f) + \frac{1}{4} \sum_{e\in E} (b_e x_e^{UE} + c_e) x_e^{UE} \qquad \text{as } c_e x_e^{UE} \geq 0 \\
&= C(f) + \frac{1}{4} C^{f^{UE}}(f^{UE})
\end{aligned}
$$

Hence we have $3/4 C(f^{UE}) \leq C(f)$.

Minimizing over admissible flow f ends the proof.

Proof

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&\leq \sum_{e\in E} \Big[(b_e x_e + c_e) x_e + \frac{1}{4} b_e (x_e^{UE})^2 \Big] && \text{as } (x_e - x_e^{UE}/2)^2 \geq 0 \\
&\leq C(f) + \frac{1}{4} \sum_{e\in E} (b_e x_e^{UE} + c_e) x_e^{UE} && \text{as } c_e x_e^{UE} \geq 0 \\
&= C(f) + \frac{1}{4} C^{f^{UE}}(f^{UE})
\end{aligned}
$$

Hence we have $3/4 C(f^{UE}) \leq C(f)$.
Minimizing over admissible flow f ends the proof.

Proof

Let f be a feasible flow, and f^{UE} be the user equilibrium. For ease of notation we fix $x^{UE} = x(f^{UE})$, and $x = x(f)$.
By Theorem we have

$$\begin{aligned}
C(f^{UE}) &\leq C^{f^{UE}}(f) \\
&= \sum_{e \in E} \big(b_e x_e^{UE} + c_e\big) x_e \\
&\leq \sum_{e \in E} \Big[\big(b_e x_e + c_e\big) x_e + \frac{1}{4} b_e \big(x_e^{UE}\big)^2\Big] && \text{as } (x_e - x_e^{UE}/2)^2 \geq 0 \\
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By Theorem we have

$$\begin{aligned} C(f^{UE}) &\leq C^{f^{UE}}(f) \\ &= \sum_{e\in E} \big(b_e x_e^{UE} + c_e\big) x_e \\ &\leq \sum_{e\in E} \Big[\big(b_e x_e + c_e\big)x_e + \frac{1}{4} b_e \big(x_e^{UE}\big)^2\Big] \qquad \text{as } (x_e - x_e^{UE}/2)^2 \geq 0 \\ &\leq C(f) + \frac{1}{4}\sum_{e\in E}\big(b_e x_e^{UE} + c_e\big)x_e^{UE} \qquad \text{as } c_e x_e^{UE} \geq 0 \\ &= C(f) + \frac{1}{4} C^{f^{UE}}(f^{UE}) \end{aligned}$$

Hence we have $3/4 C(f^{UE}) \leq C(f)$.
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Pigou's Example

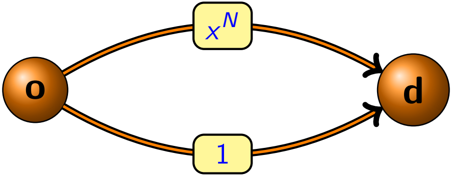

Figure: Pigou example

On a graph with two nodes: one origin, one destination, a total flow of 1, a fixed cost of 1 on one edge, and a cost of x^N on the other, where $N \in \mathbb{N}$ and x is the intensity of the flow using this edge (see Figure 1).

1. Compute the system optimum for a given N.
2. Compute the user equilibrium for a given N.
3. Compute the price of anarchy on this network when $N \to \infty$.

Exercise for next week (3.2)

Consider a (finite) directed, strongly connected, graph $G = (V, E)$. We consider K origin-destination vertex pair $\{o^k, d^k\}_{k\in[K]}$, such that there exists at least one path from o^k to d^k.
We want to find bounds on the price of anarchy, assuming that, for each arc e, $\ell_e : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, and that we have

$$x\ell_e(x) \leq \gamma L_e(x), \qquad \forall x \in \mathbb{R}^+$$

1. Recall which optimization problems solves the social optimum x^{SO} and the user equilibrium x^{UE}.
2. Let x be a feasable vector of arc-intensity. Show that $W(x) \leq C(x) \leq \gamma W(x)$.
3. Show that the price of anarchy $C(x^{UE})/C(x^{SO})$ is lower than γ.
4. If the cost per arc ℓ_e are polynomial of order at most p with non-negative coefficient, find a bound on the price of anarchy. Is this bound sharp?

Further video content

This is a research seminar by one of the expert in the domain. The first half is very interesting to get a better intuition of the concepts. The second half is more dedicated to the proof of the result presented in the talk.

https://www.youtube.com/watch?v=e3O_tMsN2t8