Operation Research and Transport
Braess's Paradox

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January 31st, 2023

What will this course be about?

- Understanding how people choose their way through a transportation network.
- having an idea on how to compute efficiently :
  - the shortest path on a network
  - the equilibrium on a network
- A practical work to compute this equilibrium on a computer
- Snapshots of other problems

Contents

1 Urban Transportation Network Analysis

2 Showcasing an example of Braess Paradox

Transportation Planning Process

1. Organization and definition
2. Base year inventory
3. Model analysis
    1. trip generation
    2. trip distribution
    3. modal split
    4. traffic assignement
4. Travel forecast
5. Network evaluation

Urban Transportation Network Analysis

Input of the analysis:

- transportation infrastructure and services (street, intersections...)
- transportation system and control policies
- demand for travel.

Two-stage analysis:

- First stage: determining the congestion, i.e. calculating the flow through each component of the network.
- Second stage : computing measure of interests according to the flow.
    - travel time and costs,
    - revenue and profit of ancilliary services,
    - welfare measures (accessibility, equity),
    - flow by-products (pollution, change in land-value)...

Why do we need a system approach?

- Some decisions could be taken according to local measures. For example, traffic lights can be timed according to data on current usual traffic at the intersection.
- However, most decisions will impact the travel time / comfort. Hence, some people will adapt their usual transit route.
- Consequently, the congestion on the network will change, changing time / comfort of other parts of the system and inducing other people to adapt their path...
- After some time these ripple effects will lessen, and the system will reach a new equilibrium.

Equilibrium in Markets

- For a given product, in a perfectly competitive market we have:
  - a production function giving the number of products companies are ready to make for a given price;
  - a demand function giving the number of product consumers are ready to buy for a given price.
- In some cases, especially in transportation, the price is not the only determinant factor. Regularity, fiability, ease of use and comfort are other determinant factor.
- In the remaining of the course we will be speaking of costs of each path, the cost factoring in all of these factors.

Nash Equilibrium : Prisonner's Dilemna

Two guys got caught while dealing chocolates. As he is missing hard evidence the judge offers them a deal.

- If both deny their implication they will get 2 months each.
- If one speaks, and the other denies, the first will get 1 month while the other will get 5 months.
- If both speak they get 4 months each.

Question: what is the equilibrium?

Nash Equilibrium

- In game theory we consider multiple agents $a \in \mathcal{A}$, each having a set of possible action $u_a \in \mathcal{U}_a$.
- Each agent earn a reward $r_a(u)$ depending on his action, as well as the other actions.
- A (pure) Nash equilibrium is a set of actions $\{u_a\}_{a \in \mathcal{A}}$, such that no player can increase his reward by changing his action if the other keeps these actions :

$$\forall a \in \mathcal{A}, \quad \forall u'_a \in \mathcal{U}_a, \qquad r_a(u'_a, u_{-a}) \leq r_a(u_a, u_{-a}).$$

- A recommendation can be followed only if it is a Nash Equilibrium.

Game Theory : a few classes

- Number of player
  - 2 (most results)
  - $n > 2$ (hard, even with 3)
  - an infinity.
- Objective
  - zero-sum game (e.g. chess)
  - cooperative: everybody shares the same objective (e.g. pandemia)
  - generic (e.g. Prisonner dilemna)

Game theory : a few definitions

Definition

A Nash equilibrium is a set of actions such that no player can unilaterally improve its pay-off by changing his action.

Definition

A Pareto efficient solution is a set of actions such that no other set of actions can strictly improve at least one player pay-off without decreasing at least another.

Definition

A social optimum is a set of actions maximizing the pay-off average.

Exercises :

- What about Prisoner's Dilemma?
- What about Zero Sum games?

Exercise: A beautiful mind

A beautiful mind : https://youtu.be/a9k4UJrCdKg

- Is the solution proposed by Nash a Nash equilibrium?
- Is the solution proposed by Nash a Pareto Optimum?
- Is the solution proposed by "Smith" a Nash equilibrium?
- Is the solution proposed by "Smith" a Pareto Optimum?
- Any other suggestion?

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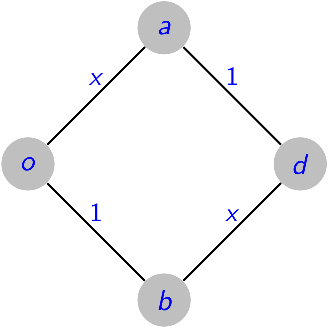
Game theory in road network

- People choose their means of transport (e.g. car versus public transport), their time of departure and their itinerary.
- Each user chooses in his own interest (mainly the shortest time / lowest cost).
- The time depends on the congestion, which means on the choice of other users.
- Hence, we are in a game framework: users interact with conflicting interests.

A very simple framework

- Consider a large group of persons who want to go from the same origin o to the destination d, at the same time, with the same car.
- We look at a very simple graph with two roads, each composed of two edges.
- The time on each edge of the road is given as a function of the number of persons taking the given edge.



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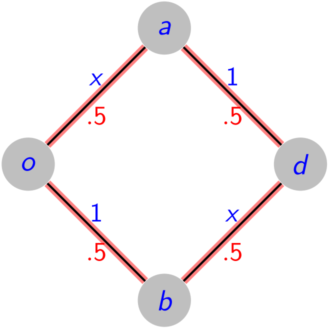

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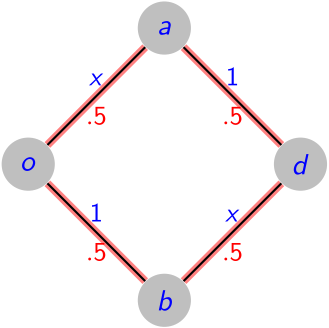
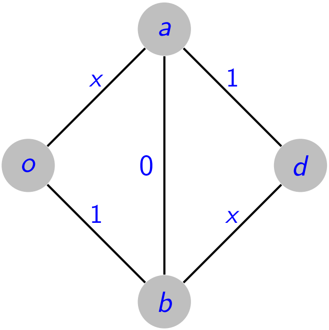


Total time : 1.5

Adding a road

- Now someone decides to construct a new, very efficient road with cost 0.
- What is the new equilibrium?



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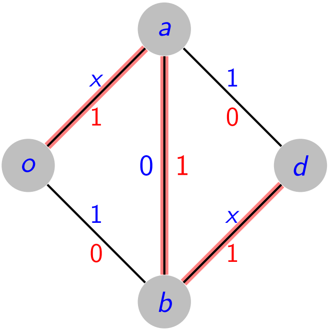
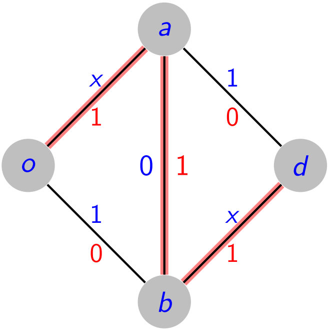


Adding a road

- Now someone decides to construct a new, very efficient road with cost 0.
- What is the new equilibrium?
- Notice that the time for every user has increased! This is the price of anarchy.



Total time : 2

Another explanation

https://www.youtube.com/watch?v=ZiauQXIKs3U (7')

And a physical demonstration:

https://www.youtube.com/watch?v=nMrYlspifuo

Definitions snapshot

On this example we can compare :

- User Equilibrium (UE), with global cost 2
- System Optimum (SO), with global cost 1.5
- price of anarchy: $4/3$.

Definition

A Wardrop (User) Equilibrium, is a repartition of flow such that no single user can improve its cost (travel time) by unilaterally changing routes.

Real case examples

- 42d Street of New York. (New York Times, 25/12/1990).
- Stuttgart 1969 (a newly built road was closed again), Seoul 2003 (6 lanes highway was turned into a park).
- New York 2009 (closed some places with success)
- In 2008, researchers found roads in Boston and NYC that should be closed to diminish traffic.
- Steinberg and Zangwill showed that Braess paradox is more or less as likely to occur as not.
- Rapoport's experiment (2009):
  - A group of 18 students is presented with the problem of repetively (40 times) choosing its road on the graph, earning money for the experiment: fastest meaning more money.
  - Then the graph is modified (either by adding the 0 cost road, or retiring it).
  - Conclusion: after a few iterations the observed repartition is close to the theoretical one with some oscillations.
  - Then tested on a bigger network.

Exercise

- Two nodes : a and b
- Two edges : (from a to b): 1 and 2
- Total number of trips: 1000
- Costs : $c_1(x_1) = 5 + 2x_1$, $c_2(x_2) = 10 + x_2$.
- Question: what is the repartition of the trips along the two edges?
- Same question with $c_1(x_1) = 15(1 + 0.15(\frac{x_1}{1000})^4)$, $c_2(x_2) = 20(1 + 0.15(\frac{x_2}{3000})^4)$?

Another Nash Equilibrium: Split or Steal

The prisonner's dilemma has been used as the final part of TV game show called "split or steal".

The rules :

- The two remaining contestants have a certain amount of money M.
- They each have to choose "split" or "steal"
- If both "split" they each get half: $M/2$.
- If one "steal" while the other "split", the stealing one gets M and the other 0.
- If they both "steal" they get nothing.

Here is an example:
https://www.youtube.com/watch?v=yM38mRHY150&list=PLq4_sHebc4IWI2VQnqaKXf0YXEj88jcK0&index=5

Here is a very nice example of why reality is more complex than math: https://www.youtube.com/watch?v=S0qjK3TWZE8