# ENPC - Operations Research and Transport - 2021

You have 2 hours for the exam. Exercises are independent. Computer, phones, tablets and every connected objects are forbidden. Every note is allowed.

**Exercice 1** (2pts)**.** Consider a game where rewards (to be maximized) are given by the following table where actions of player 1 correspond to the lines, actions of player 2 to the columns, rewards being given in the order of player. For example, if player 1 play $a$, and player 2 play $c$, then player 1 gains 0 and player 2 gains 1.

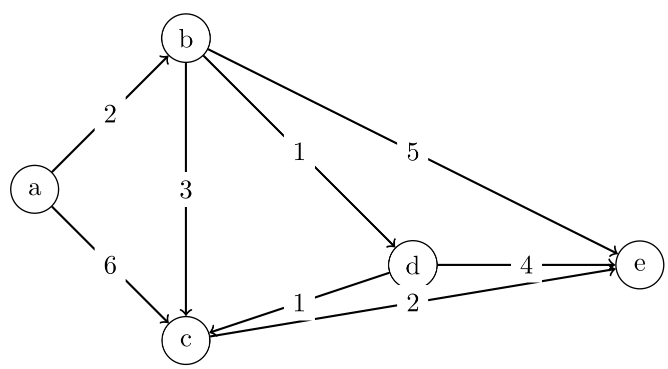
| | a | b | c |
|---|---|---|---|
| a | (1,-1) | (0,0) | (0,2) |
| b | (0,5) | (5,3) | (3,2) |
| c | (-1,3) | (6,3) | (3,5) |

1. Find the Nash equilibrium(s)
2. Find the social optimum(s)
3. Find the Pareto optimum(s)

**Solution.**

- (c,c)
- (c,b)
- (c,b), (c,c)

**Exercice 2** (5pts)**.** Consider the following weighted graph.




1. (2pts) Use Dijkstra's algorithm to find the cost of the shortest path between node $a$ and node $f$. The results can be presented in a table of the labels where each column corresponds to a node of the graph, and each line to an iteration of the Dijkstra algorithm.
2. (2pts) Find a topological ordering for the graph. Use the topological ordering to compute the cost of the shortest path from $a$ to every nodes by Dynamic Programming.
3. (1pts) Give the shortest path from $a$ to $f$.

**Exercice 3** (8pts)**.** Consider a (finite) directed, strongly connected, graph $G = (V, E)$. We consider $K$ origin-destination vertex pair $\{o^k, d^k\}_{k\in[\![1,K]\!]}$. We denote by $(G, \ell, r, c)$ the *constrained* congestion game with inflow-outflow vector rate $r$ and capacity constraint vector $c$.

- $r^k$ is the intensity of the flow of users entering in $o^k$ and exiting in $d^k$;
- $\mathcal{P}_k$ is the set of all simple (i.e. without cycle) paths from $o^k$ to $d^k$, and by $\mathcal{P} = \bigcup_{k=1}^K \mathcal{P}_k$ ;
- $f_p$ the number of users taking path $p \in \mathcal{P}$ per hour (intensity);
- $f = \{f_p\}_{p\in\mathcal{P}}$ the vector of path intensity;
- $x_e = \sum_{p \ni e} f_p$ the flux of user taking the edge $e \in E$;
- $x = \{x_e\}_{e\in E}$ the vector of edge intensity;
- $x(f)$ is the vector of edge-intensity induced by the path intensity $f$;
- $\ell_e : \mathbb{R} \to \mathbb{R}^+$ the cost incurred by a given user to take edge $e$, if the edge-intensity is $x_e$;
- $L_e(x_e) := \int_0^{x_e} \ell_e(u)du$;
- $c_e$ is the maximum flow for arc $e$.

For a constrained congestion game we say that an admissible flow $(f, x)$ is a user-equilibrium if

$$\forall k \in [K], \quad \forall p, p' \in \mathcal{P}, \quad f_p > 0 \quad \Longrightarrow \quad \begin{cases} \ell_p(f) \leq \ell_{p'}(f) & \text{or} \\ \exists e \in p' \backslash p, \quad x_e(f) = c_e \end{cases}$$

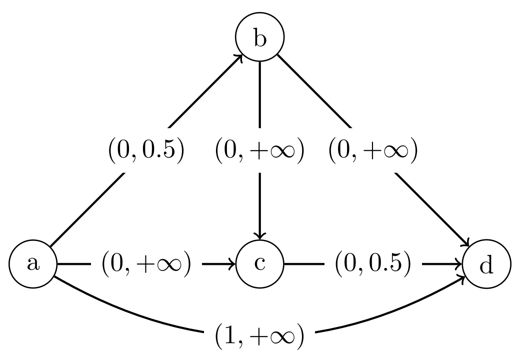



Figure 1: A simple example, each edge carry the pair $(\ell, c)$ with the cost function as the first argument (constant here), and the capacity as the second argument. The only origin destination pair is $a - d$ with a rate of 1.

1. (1pts) Justify the definition of a user-equilibrium in this setting.
2. (1.5pts) On the example shown in Figure 1, find an optimal flow. Show the existence of a user equilibrium with cost 0.5. Comment on the price of anarchy of this example.
3. (2.5pts) Suggest an optimization problem of the form

$$(CCG) \quad \min_f \quad W(x(f)) \qquad \text{s.t.} \quad f \in X$$

   where $X$ is a polyhedron given by linear constraints, such that its solution is a user equilibrium for the constrained congestion game. Prove that the solution is indeed a user equilibrium.

4. (2pts) Let $x$ and $\tilde{x}$ be admissible arc-intensity flow for $(G, \ell, r, c)$, such that

$$\sum_{e\in E} \ell_e(x_e)\tilde{x}_e < \sum_{e\in E} \ell_e(x_e)x_e.$$

Show that $f$ such that $x = x(f)$ is not an optimal solution of $(CCG)$.

5. (1pts) Show that there exists a user equilibrium $x^{UE}$ such that, for all admissible arc-intensity $\tilde{x}$,

$$\sum_{e\in E} \ell_e(x_e^{UE})x_e^{UE} \leq \sum_{e\in E} \ell_e(x_e^{UE})\tilde{x}_e.$$

**Solution.**

1. Nobody can take a strictly better path than the one he is on: other path are either more costly or saturated.

2. If 0.5 go on $a-b-c-d$, then the other 0.5 has to go on $a-d$, incurring a social cost of 0.5 while all other paths are saturated. The optimal flow consists, for example, in splitting half on $a-b-d$ and half on $a-c-d$ for a social cost of 0. Hence the "price of anarchy" (the notion is harder to define in this constrained setting) is unbounded, even for constant cost function. Indeed the cost function are actually not constant at all : they jump from 0 to $+\infty$ at 0.5 on two edges.

3. The user equilibrium problem naturally extends to

$$\begin{aligned} \min_f \quad & W(x(f)) \\ \text{s.t.} \quad & x_e = \sum_{p\ni e} f_p \\ & r_k = \sum_{p\in\mathcal{P}_k} f_p \\ & f_p \geq 0 \\ & x(f) \leq c \end{aligned}$$

Following the proof of Theorem 3.1 we introduce the Lagrangian

$$\mathcal{L}(f,\lambda,\mu,\gamma) := \mathcal{L}(f,\lambda,\mu) := W(f) + \sum_{k=1}^{K} \lambda_k \Big(r_k - \sum_{p\in\mathcal{P}_k} f_p\Big) + \sum_{p\in\mathcal{P}} \mu_p f_p + \sum_{e\in E} \gamma_e(x_e(f) - c_e)$$

leading to the following KKT conditions

$$\begin{cases} \ell_p(f) - \lambda_k + \mu_p + \sum_{e\in p} \gamma_e = 0 & \forall p \in \mathcal{P}_k, \forall k \in [\![1,K]\!] \\ r_k - \sum_{p\in\mathcal{P}_k} f_p = 0 & \forall k \in [\![1,K]\!] \\ \mu_p = 0 \text{ or } f_p = 0 & \forall p \in \mathcal{P} \\ \mu_p \leq 0, f_p \geq 0 & \forall p \in \mathcal{P} \\ \gamma_e = 0 \text{ or } x_e(f) = c_e & \forall e \in E \\ \gamma_e \geq 0 & \forall e \in E \end{cases}$$

the result follows : if either $f_p = 0$ or $x_e(f) = c_e$ for some $e \in p'$, then $\ell_p(f)$ can be greater than $\lambda_k$ which is still the shortest path.

4. The set of admissible flow is convex, thus $x_\lambda = (1-\lambda)x + \lambda\tilde{x}$ is admissible. And we have

$$\begin{aligned} W(x_\lambda) &= W(x + \lambda(\tilde{x} - x)) \\ &= W(x) + \lambda\Big[\sum_{e\in E} \ell_e(x_e)\tilde{x}_e - \sum_{e\in E} \ell_e(x_e)x_e\Big] + o(\lambda) \end{aligned}$$

Meaning that, for $\lambda > 0$ small enough, $x_\lambda$ yields a better $W$ than $x$.

5. Problem $(CCG)$ is convex and $X$ is a compact polyhedron. There exists an optimal solution to $(CCG)$. Constraint are qualified and this optimal solution satisfy the KKT solution, thus being a user equilibrium. Finally, by the previous question, $x^{UE}$ being an optimal solution, we can't have the strict inequality hence the result.

**Exercice 4** (5 points)**.** Consider the following problem

$$\begin{aligned} \min_{x\in\mathbb{R}^2} \quad & \frac{1}{2}(x_1-\frac{1}{2})^2+\frac{1}{2}x_2^2 \\ \text{s.t.} \quad & 0 \leq x_1 \leq 1 \\ & 0 \leq x_2 \leq 1 \end{aligned}$$

A plot of the different points used during the iterations would be usefull.

1. (2pts) Starting at $x^{(0)} = (1,1)$ compute the first iteration (i.e. find $x^{(1)}$) of the Frank Wolfe algorithm (with optimal step size).
2. (1pts) Give the lower and upper bound obtained by this first iteration.
3. (2pts) Do a second iteration. Have we reached the optimum?

**Solution.**

1. We have $f(x) = \frac{1}{2}(x_1-\frac{1}{2})^2+\frac{1}{2}x_2^2$, and $\nabla f(x) = \begin{pmatrix} x_1 - \frac{1}{2} \\ x_2 \end{pmatrix}$. Thus, at iteration $k=0$, $\nabla f(x^{(0)}) = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$ we have to solve $\min_{y\in X} \frac{1}{2}y_1 + y_2$, leading to $y^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and $d^{(0)}) = y^{(0)} - x^{(0)} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$. The line-search problem reads
$$\min_{t\in[0,1]} f(1-t,1-t) = \frac{1}{2}(\frac{1}{2}-t)^2 + \frac{1}{2}(1-t)^2$$
by derivation the minimum is attained for $t$ satisfying $-(\frac{1}{2}-t)-(1-t) = 0$, thus $t^{(0)} = 3/4$, and $x^{(1)} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$.
2. $UB = f(x^{(1)}) = \frac{1}{16}$.
$LB = f(x^{(0)}) - \nabla f(x^{(0)})^\top x^{(0)} + \min_{y\in X} \nabla f(x^{(0)})^\top y = \frac{5}{8} - \frac{3}{2} + 0 = -\frac{7}{8}$.
3. For $k=1$, we have $\nabla f(x^{(1)}) = \begin{pmatrix} -\frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$ leading to $y^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $d^{(1)} = y^{(1)} - x^{(1)} = \begin{pmatrix} \frac{3}{4} \\ -\frac{1}{4} \end{pmatrix}$. The line search problem reads
$$\min_{t\in[0,1]} \quad f(\frac{1}{4}+\frac{3t}{4}, \frac{1}{4}-\frac{t}{4}) = \frac{1}{2}(-\frac{1}{4}+\frac{3t}{4})^2 + \frac{1}{2}(\frac{1}{4}-\frac{t}{4})^2 = \frac{1}{32}[(3t-1)^2+(t-1)^2]$$
attained for $t$ solving $3(3t-1)+(t-1) = 0$, that is $t^{(1)} = \frac{2}{5}$, and $x^{(2)} = \begin{pmatrix} \frac{11}{20} \\ \frac{3}{20} \end{pmatrix}$. This is not the optimal solution (which is $\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$).