ENPC - Operations Research and Transport - 2019

You have 2.5 hours for the exam. Exercises are independent. Computer, phones, tablets and every connected objects are forbidden. Every note is allowed.

Exercice 1 (7pts)**.** Consider a game where rewards (to be maximized) are given by the following table where actions of player 1 correspond to the lines, actions of player 2 to the columns, rewards being given in the order of player.

	a	b
a	(-5,-5)	(1,-1)
b	(-1,1)	(0,0)

1. Find the Nash equilibrium(s), social optimum(s) and Pareto optimum(s)
2. We now want to consider random strategies. More precisely we consider that player one play a with probability p_1 and player 2 play a with probability p_2 (independently of the action of 1). We assume that each wants to maximize its expected reward.
   (a) For given p_1 and p_2 what is the expected reward of player 1?
   (b) For a given p_2 what are the set of p_1 maximizing the expected reward of player 1?
   (c) Justify that, when looking for a Nash-Equilibrium, only 3 value of p_1 and p_2 should be considered, and give the reward matrix associated.
   (d) What are the Nash Equilibrium(s)? Is it better than in the original deterministic version?

Solution. 1. (1.5pt) NE : (a,b), (b,a) ; OS and Pareto : (a,b), (b,a), (b,b)

2. (a) (1pt) The reward obtained by 1 is $-5p_1p_2 + p_1(1-p_2) - p_2(1-p_1) = -5p_1p_2 + p_1 - p_2 = p_1(1-5p_2) - p_2$
   (b) (1.5pt) For $p_2 > 1/5$, the optimal p_1 is 0. For $p_2 < 1/5$ the optimal p_1 is 1. For $p_2 = 1/5$, every $p_1 \in [0, 1]$ is optimal.
   (c) (2pt) By symmetry we have the same result for p_2, hence we have

	0	0.2	1
0	(0, 0)	(-0.2, 0.2)	(-1, 1)
0.2	(0.2, -0.2)	(-0.2, -0.2)	(-1.8, -0.2)
1	(1, -1)	(-0.2, -1.8)	(-5, -5)

   (d) (1pt) The Nash Equilibrium is (0.2,0.2), with a social value of -0.4 which is worse than 0. However it is symmetric.

Exercice 2 (13pts)**.** Consider a (finite) directed, strongly connected, graph $G = (V, E)$. We consider K origin-destination vertex pair $\{o^k, d^k\}_{k\in[\![1,K]\!]}$. We denote by (G, ℓ, r) the congestion game with inflow vector r.

- r^k is the intensity of the flow of users entering in o^k and exiting in d^k;
- $\mathcal{P}_k$ is the set of all simple (i.e. without cycle) paths from o^k to d^k, and by $\mathcal{P} = \bigcup_{k=1}^{K} \mathcal{P}_k$;
- f_p the number of users taking path $p \in \mathcal{P}$ per hour (intensity);

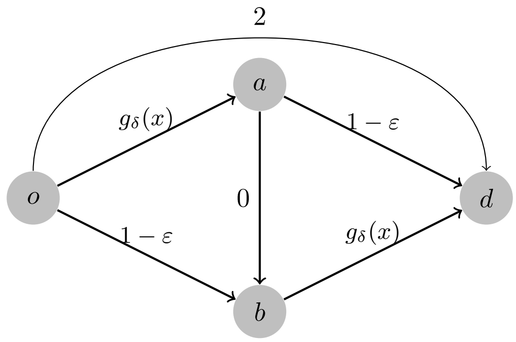

Figure 1: A graph example

- $f = \{f_p\}_{p\in\mathcal{P}}$ the vector of path intensity;
- $x_e = \sum_{p\ni e} f_p$ the flux of user taking the edge $e \in E$;
- $x = \{x_e\}_{e\in E}$ the vector of edge intensity;
- $x(f)$ is the vector of edge-intensity induced by the path intensity f;
- $\ell_e : \mathbb{R} \to \mathbb{R}^+$ the cost incurred by a given user to take edge e, if the edge-intensity is x_e;
- $L_e(x_e) := \int_0^{x_e} \ell_e(u)du$.

We say that an admissible flow f^ε, for $\varepsilon \in [0, 1[$ is a **ε-Nash Equilibrium** if

$$\forall k \in [\![1, K]\!], \quad \forall p_1, p_2 \in \mathcal{P}_k, \quad f^\varepsilon_{p_1} > 0 \quad \Longrightarrow \ell_{p_1}(f^\varepsilon) \leq (1+\varepsilon)\ell_{p_2}(f^\varepsilon).$$

We want to compare the cost of a given ε-equilibrium of (G, ℓ, r), denoted $f^{\varepsilon,r}$, with the cost of the social optimum $f^{SO,2r}$ of $(G, \ell, 2r)$, that is the same game with twice the inflows. Accordingly we denote $x^{\varepsilon,r} = x(f^{\varepsilon,r})$, and $x^{SO,2r} = x(f^{SO,2r})$. Finally, edge-loss ℓ_e are assumed to be non-negative and non-decreasing.

Both parts are largely independent.

Part I : an example

We consider, for $\varepsilon \in [0, 1[$, the congestion game (G, ℓ, r) given in Figure **??** with the unique origin destination pair $o - d$. Here, g_δ is a continuous non-decreasing function with value 0 on $] - \infty, 1 - \delta]$ and value $1 + \varepsilon$ on $[1, +\infty[$.

1. Show that a flow $f^{\varepsilon,1}$ getting 1 through $o \to a \to b \to d$, and 0 on other paths, is a ε-Nash Equilibrium of $(G, \ell, 1)$.
2. Construct an admissible flow of $(G, \ell, 2)$ of cost $4\delta + 2(1-\varepsilon)(1-\delta)$.
3. Show that the social optimum of $(G, \ell, 2)$ can be found by solving an unidimensional optimization problem, and propose an adapted optimization algorithm.

Solution.
1. (1pt) There are 4 possible paths : $o - d$, $o - a - d$, $o - a - b - d$, $o - b - d$. For $f^{\varepsilon,1}$ their cost is 2, $1 + \varepsilon + 1 - \varepsilon = 2$, $2 + 2\varepsilon$ and 2. Thus $f^{\varepsilon,1}$ is an ε-Nash Equilibrium.
2. (1pt) We put δ on $o - d$, $1 - \delta/2$ on $o - a - d$ and $1 - \delta/2$ on $o - b - d$.
3. (2.5pts) The global cost is

$$2f_1 + (f_2 + f_3)g_\delta(f_2 + f_3) + (1-\varepsilon)f_2 + (1-\varepsilon)f_3 + g_\delta(f_3)$$

we can improve the cost of any admissible flow by shifting from path 3 to path 4 (as g_δ is increasing), thus an optimal flow have $f_3 = 0$. By monotonicity, an optimal solution have $f_2 = f_4$, and as $f_1 + f_2 + f_3 + f_4 = 2$ we reduce the problem to

$$\min_{f_4 \in \ [0,1]} 2(2 - 2f_4) + 2f_4 g_\delta(f_4)$$

which can be further reduced to

$$\min_{f_4 \in \ [1-\delta,1]} 2(2 - 2f_4) + 2f_4 g_\delta(f_4)$$

Part II : bounding the cost of ε-Nash Equilibrium

We construct new loss functions $\bar{\ell}_e(x)$ given by

$$\bar{\ell}_e(x) = \begin{cases} \ell_e(x^{\varepsilon,r}) & \text{if } x \leq x^{\varepsilon,r} \\ \ell_e(x) & \text{else} \end{cases}$$

Accordingly we denote $\bar{\ell}_p(f) = \sum_{e \in p} \bar{\ell}_e(x_e(f))$ and

$$C(x) = \sum_{e \in E} x_e \ell_e(x_e) \qquad \text{and} \qquad \bar{C}(x) = \sum_{e \in E} x_e \bar{\ell}_e(x_e).$$

For $k \in [\![1, K]\!]$, denote $\lambda_k(x)$ the minimum cost of an o_k-d_k-path with costs given by edge-intensity vector x.

4. Give an interpretation of an ε-Nash Equilibrium. What happens if $\varepsilon = 0$?
5. Show that $C(x^{\varepsilon,r}) \leq (1 + \varepsilon) \sum_{k=1}^{K} r_k \lambda_k(x^{\varepsilon,r})$.
6. Show that, for any $x \in \mathbb{R}_+^{|E|}$, $x_e(\bar{\ell}_e(x_e) - \ell_e(x_e)) \leq x_e^{\varepsilon,r} \ell_e(x_e^{\varepsilon,r})$.
7. Deduce that, $\bar{C}(x^{SO,2r}) - C(x^{SO,2r}) \leq C(x^{\varepsilon,r})$.
8. Show that, for all $p \in \mathcal{P}_k$, $\bar{\ell}_p(f^{SO,2r}) \geq \lambda_k(x^{\varepsilon,r})$.
9. Show that
$$\sum_{p \in \mathcal{P}} \bar{\ell}_p(f^{SO,2r}) f_p^{SO,2r} \geq \frac{2}{1 + \varepsilon} C(x^{\varepsilon,r})$$
10. Find a constant K_ε such that $C(x^{\varepsilon,r}) \leq K_\varepsilon C(x^{SO,2r})$.
11. Using part I show that this bound is tight.

Solution.

1. (0.5pts) An ε-Nash equilibrium is a flux such that each user can win at most ε by changing trajectory with fixed cost. If $\varepsilon = 0$ we recover the Wardrop equilibrium.
2. (1pt) $f^{\varepsilon,r}$ is a ε-Nash equilibrium, thus for every $p \in \mathcal{P}_k$ we have $f_p^{\varepsilon,r} \ell_p(f^{\varepsilon,r}) \leq (1 + \varepsilon) f_p^{\varepsilon,r} \lambda_k(x(f^{\varepsilon,r}))$, and summing over all $p \in \mathcal{P}$ yields the result.
3. (1pt) $\bar{\ell}_e(x_e) - \ell_e(x_e)$ is null if $x_e \geq x_e^{\varepsilon,r}$, and equal to $\ell_e(x_e^{\varepsilon,r}) - \ell_e(x_e) \leq \ell_e(x_e^{\varepsilon,r})$ otherwise. Multiplying by x_e we have the result both for $x_e \geq x_e^{\varepsilon,r}$ and for $x_e \leq x_e^{\varepsilon,r}$.
4. (1pt)
$$\begin{aligned} \bar{C}(x^{SO,2r}) - C(x^{SO,2r}) &= \sum_{e \in E} x_e^{SO,2r} (\bar{\ell}_e(x_e^{SO,2r}) - \ell_e(x_e^{SO,2r})) \\ &\leq \sum_{e \in E} x_e^{\varepsilon,r} \ell_e(x_e^{\varepsilon,r}) \\ &= C(x^{\varepsilon,r}) \end{aligned}$$

5. (1pt) We have $\bar{\ell}_p(0) \geq \lambda_k(x^{\varepsilon,r})$, and as $\bar{\ell}_p$ is non-decreasing we get $\bar{\ell}_p(f^{SO,2r}) \geq \lambda_k(x^{\varepsilon,r})$.

6. (1.5pts)

$$\begin{aligned}
\sum_{p\in\mathcal{P}} \bar{\ell}_p(f^{SO,2r})f_p^{SO,2r} &\geq \sum_k \sum_{p\in\mathcal{P}_k} \lambda_k(x^{\varepsilon,r})f_p^{SO,2r} && \text{by previous question}\\
&= \sum_k \lambda_k(x^{\varepsilon,r})r_k\\
&\geq \frac{2}{1+\varepsilon}C(f^{\varepsilon,r}) && \text{by question \textbf{??}}
\end{aligned}$$

7. (1.5pts) We have

$$\begin{aligned}
C(x^{SO,2r}) &\geq \sum_{p\in\mathcal{P}} \bar{\ell}_p(f^{SO,2r})f_p^{SO,2r} - C(x^{\varepsilon,r})\\
&\geq \frac{2}{1+\varepsilon}C(x^{\varepsilon,r}) - C(x^{\varepsilon,r})\\
&= \frac{1-\varepsilon}{1+\varepsilon}C(x^{\varepsilon,r})
\end{aligned}$$

8. (1pt) In the example of part I we have $C(x^{\varepsilon,1}) = 2 + 2\varepsilon$, and an admissible flow for the double rate with cost $2\delta + (1-\varepsilon)(1-\delta)$. Letting δ goes to zero yields the result.

9. (1.5pts) Consider $p \in \mathcal{P}^k$. Then $\ell_p(f^{UE,r}) = c_k$. Furthermore,

$$\bar{\ell}_p(f^{SO,2r}) = \sum_{e\in E} \bar{\ell}_e(x_e(f^{SO,2r})) \geq \sum_{e\in E} \ell_e(x_e^{UE,r}) = c_k$$

where the inequality comes from monotonicity of ℓ_e, and definition of $\bar{\ell}_e$.

10. (0.5pts)

$$C(x) = \sum_{f\in\mathcal{P}} f_p\ell_p(f) \qquad \text{and} \qquad \bar{C}(x) = \sum_{f\in\mathcal{P}} f_p\bar{\ell}_p(f).$$

11. (2pts)

$$\begin{aligned}
\bar{C}(f^{SO,2r}) &= \sum_{k=1}^{K} \sum_{p\in\mathcal{P}_k} f_p^{SO,2r}\bar{\ell}_p(f^{SO,2r})\\
&\geq \sum_{k=1}^{K} c_k \sum_{p\in\mathcal{P}_k} f_p^{SO,2r}\\
&= \sum_{k=1}^{K} 2c_k r^k\\
&= 2C(f^{UE,r})
\end{aligned}$$

12. (1pts) Combining previous results we have

$$2C(f^{UE,r}) \leq \bar{C}(x^{SO,2r}) \leq C(x^{UE,r}) + C(x^{SO,2r}),$$

which give the result, that can be interpreted as "optimizing flux cannot allow more than twice the inflows rates without increasing global cost".

Exercice 3 (1pt)**.** Present two operation research problems encountered by Air France.