# ENPC - Operations Research and Transport - 2018

You have 2.5 hours for the exam. Exercises are independent. Computer, phones, tablets and every connected objects are forbidden. Every note is allowed.

**Exercice 1** (2pts)**.** *Consider a game where rewards (to be maximized) are given by the following table where actions of player* 1 *correspond to the lines, actions of player* 2 *to the columns, rewards being given in the order of player. For example, if player* 1 *play* a*, and player* 2 *play* c*, then player* 1 *gains* 2 *and player* 2 *gains* 3*.*

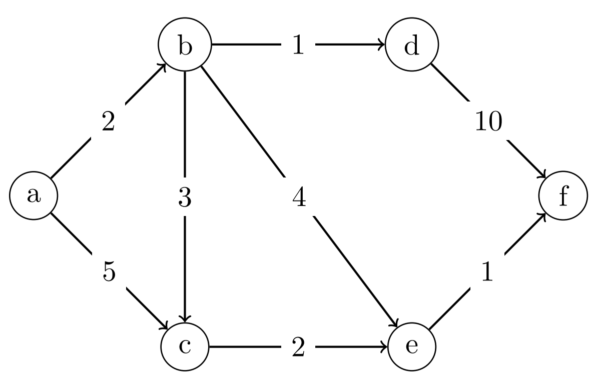
| | a | b | c |
|---|---|---|---|
| a | (7,1) | (0,0) | (2,3) |
| b | (-1,2) | (2,3) | (3,2) |
| c | (-1,4) | (1,3) | (1,7) |

1. *Find the Nash equilibrium(s)*
2. *Find the social optimum(s)*
3. *Find the Pareto optimum(s)*

**Solution.** 1. *(0.5pt) NE :* $(b, b)$

2. *(0.5pt) SO :* $(a, a)$, $(c, c)$
3. *(1.25 pt) Pareto :* $(a, a)$, $(a, c)$, $(b, b)$, $(b, c)$ $(c, c)$,

**Exercice 2** (5pts)**.** *Consider the following weighted graph.*




1. *Use Dijkstra's algorithm to find the cost of the shortest path between node* $a$ *and node* $f$*. The results can be presented in a table of the labels where each column corresponds to a node of the graph, and each line to an iteration of the Dijkstra algorithm. Note the order in which the nodes are treated.*

| a | b | c | d | e | f |
|---|---|---|---|---|---|
| 20 | 4 | 7 | 6 | 0 | 0 |

2. *We have the following heuristic $h$ giving an estimate of the distance between a given node and $f$.*

   *Apply the $A^*$ algorithm using this heuristic. Note the order of nodes treated. Comment.*

**Solution.** 1. *We have (2 pt)*

| $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|
| (0) | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| 0 | (2) | (5) | $\infty$ | $\infty$ | $\infty$ |
| 0 | 2 | (5) | (3) | (6) | $\infty$ |
| 0 | 2 | (5) | 3 | (6) | (13) |
| 0 | 2 | 5 | 3 | (6) | (13) |
| 0 | 2 | 5 | 3 | 6 | (7) |
| 0 | 2 | 5 | 3 | 6 | 7 |

*Hence the shortest path from $a$ to $f$ as cost 7. The nodes are treated in the following order: a-b-d-c-e-$f$.*

2. *(3pt)*

   *We compute the label $\lambda$ in the following table*

| $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|
| (20) | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| 20 | (6) | (12) | $\infty$ | $\infty$ | $\infty$ |
| 20 | 6 | (12) | (9) | (6) | $\infty$ |
| 20 | 6 | (12) | (9) | 6 | (7) |
| 20 | 6 | (12) | (9) | 6 | 7 |

*Which gives the distance and shortest path in only 4 iterations (a-b-e-$f$) instead of 6.*

**Exercice 3** (8pts)**.** *Consider a (finite) directed, strongly connected, graph $G = (V, E)$. We consider $K$ origin-destination vertex pair $\{o^k, d^k\}_{k\in[\![1,K]\!]}$. We denote by $(G, \ell, r)$ the congestion game where*

- *$r^k$ is the intensity of the flow of users entering in $o^k$ and exiting in $d^k$;*
- *$\mathcal{P}_k$ is the set of all simple (i.e. without cycle) paths from $o^k$ to $d^k$, and by $\mathcal{P} = \bigcup_{k=1}^{K} \mathcal{P}_k$ ;*
- *$f_p$ the number of users taking path $p \in \mathcal{P}$ per hour (intensity);*
- *$f = \{f_p\}_{p\in\mathcal{P}}$ the vector of path intensity;*
- *$x_e = \sum_{p\ni e} f_p$ the flux of user taking the arc $e \in E$;*
- *$x = \{x_e\}_{e\in E}$ the vector of arc intensity;*
- *$x(f)$ is the vector of edge-intensity induced by the path intensity $f$;*
- *$\ell_e : \mathbb{R} \to \mathbb{R}^+$ the cost incurred by a given user to take edge $e$, if the edge-intensity is $x_e$;*

- $L_e(x_e) := \int_0^{x_e} \ell_e(u)du$.

*We want to compare the cost of the user equilibrium of $(G, \ell, r)$, denoted $f^{UE,r}$, with the cost of the social optimum $f^{SO,2r}$ of $(G, \ell, 2r)$, that is the same game with twice the inflows. Accordingly we denote $x^{UE,r} = x(f^{UE,r})$, and $x^{SO,2r} = x(f^{SO,2r})$. Finally, edge-loss $\ell_e$ are assumed to be non-negative and non-decreasing.*

*We construct new loss functions $\bar{\ell}_e(x)$ given by*

$$\bar{\ell}_e(x) = \begin{cases} \ell_e(x^{UE,r}) & \text{if } x \leq x^{UE,r} \\ \ell_e(x) & \text{else} \end{cases}$$

*Accordingly we denote $\bar{\ell}_p(f) = \sum_{e \in p} \bar{\ell}_e(x_e(f))$ and*

$$C(x) = \sum_{e \in E} x_e \ell_e(x_e) \qquad \text{and} \qquad \bar{C}(x) = \sum_{e \in E} x_e \bar{\ell}_e(x_e).$$

1. *Justify that for all $k \in [\![1, K]\!]$, there exists $c_k \in \mathbb{R}_+$ such that for all path $p \in \mathcal{P}_k$,*

$$f_p^{UE,r} > 0 \;\Rightarrow\; \ell_p(f^{UE,r}) = c_k.$$

2. *Show that, for any $x \in \mathbb{R}_+^{|E|}$, $C(x) \leq \bar{C}(x)$, and that $C(x^{UE,r}) = \bar{C}(x^{UE,r})$.*
3. *Show that, for any $x \in \mathbb{R}_+^{|E|}$, $x_e(\bar{\ell}_e(x_e) - \ell_e(x_e)) \leq x_e^{UE,r} \ell_e(x_e^{UE,r})$.*
4. *Deduce that, $\bar{C}(x^{SO,2r}) - C(x^{SO,2r}) \leq C(x^{UE,r})$.*
5. *On the other hand, show that, for every path $p \in \mathcal{P}_k$, $\bar{\ell}_p(f^{SO,2r}) \geq c_k$.*
6. *Write $C$ and $\bar{C}$ as function of $f$ instead of $x$ (we keep the same notation).*
7. *Deduce that, $\bar{C}(f^{SO,2r}) \geq 2C(f^{UE,r})$.*
8. *Finally, show that, $C(f^{UE,r}) \leq C(f^{SO,2r})$. Give an interpretation of this result.*

**Solution.**

1. *(0.5pts) $f^{UE,r}$ is a Wardrop equilibrium, thus by definition the cost of all used path is the same.*
2. *(0.5pts) As $\ell_e$ are non decreasing, $\bar{\ell}_e \geq \ell_e$, multiplying by $x_e \geq 0$ and summing gives the result. Equality is obvious.*
3. *(1pts) $\bar{\ell}_e(x_e) - \ell_e(x_e)$ is null if $x_e \geq x_e^{UE,r}$, and equal to $\ell_e(x_e^{UE,r}) - \ell_e(x_e) \leq \ell_e(x_e^{UE,r})$ otherwise. Multiplying by $x_e$ we have the result both for $x_e \geq x_e^{UE,r}$ and for $x_e \leq x_e^{UE,r}$.*
4. *(1pts)*

$$\begin{aligned} \bar{C}(x^{SO,2r}) - C(x^{SO,2r}) &= \sum_{e \in E} x_e^{SO,2r} (\bar{\ell}_e(x_e^{SO,2r}) - \ell_e(x_e^{SO,2r})) \\ &\leq \sum_{e \in E} x_e^{UE,r} \ell_e(x_e^{UE,r}) \\ &= C(x^{UE,r}) \end{aligned}$$

5. *(1.5pts) Consider $p \in \mathcal{P}^k$. Then $\ell_p(f^{UE,r}) = c_k$. Furthermore,*

$$\bar{\ell}_p(f^{SO,2r}) = \sum_{e \in E} \bar{\ell}_e(x_e(f^{SO,2r})) \geq \sum_{e \in E} \ell_e(x_e^{UE,r}) = c_k$$

*where the inequality comes from monotonicity of $\ell_e$, and definition of $\bar{\ell}_e$.*

6. *(0.5pts)*

$$C(x) = \sum_{f \in \mathcal{P}} f_p \ell_p(f) \qquad and \qquad \bar{C}(x) = \sum_{f \in \mathcal{P}} f_p \bar{\ell}_p(f).$$

7. *(2pts)*

$$\begin{aligned} \bar{C}(f^{SO,2r}) &= \sum_{k=1}^{K} \sum_{p \in \mathcal{P}_k} f_p^{SO,2r} \bar{\ell}_p(f^{SO,2r}) \\ &\geq \sum_{k=1}^{K} c_k \sum_{p \in \mathcal{P}_k} f_p^{SO,2r} \\ &= \sum_{k=1}^{K} 2c_k r^k \\ &= 2C(f^{UE,r}) \end{aligned}$$

8. *(1pts) Combining previous results we have*

$$2C(f^{UE,r}) \leq \bar{C}(x^{SO,2r}) \leq C(x^{UE,r}) + C(x^{SO,2r}),$$

*which give the result, that can be interpreted as "optimizing flux cannot allow more than twice the inflows rates without increasing global cost".*

**Exercice 4** (7pts)**.** *Consider the function $f(x_1, x_2) = 4x_1^4 - 2x_1 + x_2^2 - x_2 + 2$, and the set*

$$X = \left\{ x \in \mathbb{R}_+^2 \quad | \quad 2x_2 + x_1 \leq 2 \right\}$$

*and $x^0 = (0,0)$. A scheme of $X$ representing the iteration and search direction of the algorithm might be helpful.*

1. *Justify that $X$ is polyhedral and find its extreme points.*
2. *Compute $\nabla f$*
3. *Justify that this problem can be solved by Frank-Wolfe (aka conditional gradient) algorithm.*
4. *Find the descent direction $d^0$ of the Frank-Wolfe algorithm starting from $x_0$.*
   *(hint : use the extreme points of $X$).*
5. *Find the optimal step $t^0$ of the first step of Frank-Wolfe algorithm. What is the new point $x^1$?*
6. *What is the upper and lower bound obtained along this first iteration?*
7. *Find the descent direction $d^1$ of the second step of Frank-Wolfe algorithm.*

8. *Write the unidimensional optimisation problem that would determine the next optimal step $t^1$ (do not solve it).*

9. *Compute the lower bound associated to the second step of the algorithm.*

**Solution.** 1. *(0.5pts)* $(0,0)$, $(0,1)$ *and* $(2,0)$

2. *(0.5pts)* $\nabla(f) = \begin{pmatrix} 16x_1^3 - 2 \\ 2x_2 - 1 \end{pmatrix}$

3. *(0.5pts)* $\nabla^2 f(x) = \begin{pmatrix} 42x_1^2 & 0 \\ 0 & 2 \end{pmatrix} \succeq 0$ *hence $f$ is convex, and $X$ is polyhedral and bounded.*

4. *(1pts)* $\min_{y \in X} -2y_1 - 1y_2$. $-2*0-1*1 > -2*2-1*0$, *hence the optimal solution is* $y^0 = (2,0)$. *And the optimal direction is* $d^0 = y^0 - x^0 = (2,0)$.

5. *(1pts)* $\min_{t\in[0,1]} 2^6t^4 - 2^2t$. *By derivating this objective function we obtain that the optimal step is* $t^0 = 1/4$, *and* $x^1 = x^0 + t^0d^0 = (1/2,0)$.

6. *(1pts) The upper bound is* $f(x^1) = 5/4$. *The lower bound is* $(\nabla(f)(x^0))^T(y^0 - x^0) + f(x^0) = -2$.

7. *(1pts) To find the direction $d^1$ we need to solve* $\min_{y\in X} \nabla(f)(x^1)^T y$. *The solution being an extreme point of $X$ that is neither $(0,0)$ nor $(2,0)$, we have $y^1 = (1,0)$. Thus* $d^1 = (1,-1/2)$.

8. *(0.5pts) Finding the optimal step require to solve*

$$\min_{t\in[0,1]} 4(1/2-t)^4 - 2(1/2-t) + t^2 - t$$

9. *(1pts) The lower bound is given by*

$$(\nabla(f)(x^1))^T(y^1 - x^1) + f(x^1) = 0*(-1/2) - 1*1 + 5/4 = 1/4$$

**Exercice 5** (Bonus). *According to Yuso, why is the package transport problem more complicated than the taxi problem? Why are Yuso solving shortest path problem for?*