ENPC - Operations Research and Transport - 2017

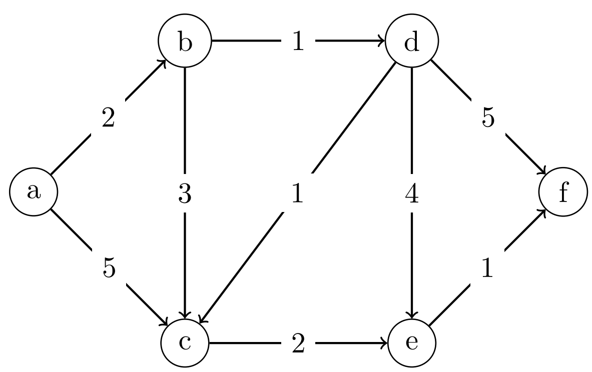
You have 2 hours for the exam. Exercises are independent. Computer, phones, tablets and every connected objects are forbidden. Every note is allowed.

Exercice 1 (2pts)**.** *Consider a game where rewards (to be maximized) are given by the following table where actions of player* 1 *correspond to the lines, actions of player* 2 *to the columns, rewards being given in the order of player. For example, if player* 1 *play* a*, and player* 2 *play* c*, then player* 1 *gains* 0 *and player* 2 *gains* 1*.*

	a	b	c	d
a	(1,-1)	(0,0)	(0,1)	(-1,4)
b	(-1,2)	(2,3)	(3,2)	(-2,3)

1. *Find the Nash equilibrium(s)*
2. *Find the social optimum(s)*
3. *Find the Pareto optimum(s)*

Exercice 2 (5pts)**.** *Consider the following weighted graph.*



1. *Use Dijkstra's algorithm to find the cost of the shortest path between node* a *and node* f*. The results can be presented in a table of the labels where each column corresponds to a node of the graph, and each line to an iteration of the Dijkstra algorithm.*
2. *Find a topological ordering for the graph. Use the topological ordering to compute the cost of the shortest path from* a *to every nodes by Dynamic Programming.*
3. *Give the shortest path from* a *to* f*.*

Exercice 3 (6pts)**.** *Consider a (finite) directed, strongly connected, graph $G = (V, E)$. We consider K origin-destination vertex pair $\{o^k, d^k\}_{k\in[\![1,K]\!]}$, such that there exists at least one path from o^k to d^k. Let denote by*

1. *r^k the intensity of the flow of users entering in o^k and exiting in d^k;*
2. *$\mathcal{P}_k$ the set of all simple (i.e. without cycle) paths from o^k to d^k, and by $\mathcal{P} = \bigcup_{k=1}^K \mathcal{P}_k$;*
3. *f_p the number of users taking path $p \in \mathcal{P}$ per hour (intensity);*
4. *$f = \{f_p\}_{p\in\mathcal{P}}$ the vector of path intensity;*
5. *$x_a = \sum_{p\ni e} f_p$ the flux of user taking the arc $e \in E$;*
6. *$x = \{x_e\}_{e\in E}$ the vector of arc intensity;*
7. *$\ell_e : \mathbb{R} \to \mathbb{R}^+$ the cost incurred by a given user to take edge e, if the edge-intensity is x_e;*
8. *$L_e(x_e) := \int_0^{x_e} \ell_e(u)du$.*

We want to find bounds on the price of anarchy, assuming that, for each arc e, $\ell_e : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, and that we have

$$x\ell_e(x) \leq \gamma L_e(x), \qquad \forall x \in \mathbb{R}^+$$

1. *Recall which optimization problems solves the social optimum x^{SO} and the user equilibrium x^{UE}. We will denote $W(x)$ the objectif function of the user equilibrium problem and $C(x)$ the objective function of the social optimum problem.*
2. *Let x be a feasable vector of arc-intensity. Show that $W(x) \leq C(x) \leq \gamma W(x)$.*
3. *Show that the price of anarchy $C(x^{UE})/C(x^{SO})$ is lower than γ.*
4. *If the cost per arc ℓ_e are polynomial of order at most p with non-negative coefficient, find a bound on the price of anarchy. Is this bound sharp?*

Exercice 4 (bonus)**.** *Give three ideas that allow to speed-up the shortest path algorithm in Google Maps.*