Robust Optimization: Adaptive Case

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Two-stage robust optimization problem

Recall that a generic robust optimization problem reads as follows:

$$\min_{x\in\mathbb{R}^p} \quad \max_{\xi\in R} \quad J(x,\xi) + \mathbb{I}_{g(x,\xi)\leq 0}$$

This problem is static or one-stage because the decision maker has to choose x before knowing the realization of ξ.

In the two-stage or adaptive case, the decision maker can choose a recourse decision y after observing the realization of ξ:

$$\min_{x\in\mathbb{R}^p} \quad \max_{\xi\in R} \quad \min_{y\in\mathbb{R}^q} \quad J(x,\xi,y) + \mathbb{I}_{g(x,\xi,y)\leq 0}$$

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Example

Solve the following example by eliminating the recourse variables:

$$
\begin{aligned}
\min_{x\in\mathbb{R}, y\in\mathbb{R}^{[0,1]}} \quad & x \\
\text{s.t.} \quad & x - y_\xi \leq -\xi && \forall \xi \in [0,1] \\
& -x + y_\xi \leq \xi + \frac{1}{2} && \forall \xi \in [0,1] \\
& y_\xi \geq 1 && \forall \xi \in [0,1]
\end{aligned}
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The constraints are equivalent to:

$$\begin{cases} y_\xi & \geq & x + \xi \\ y_\xi & \leq & x + \xi + \frac{1}{2} \\ y_\xi & \geq & 1 \end{cases} \iff \begin{cases} \max\left\{1, \xi + \frac{1}{2}\right\} & \leq & y_\xi \leq 1 + \xi \\ 1 & \leq & x + \xi + \frac{1}{2} \end{cases}$$

Thus, the optimal value is $\frac{1}{2}$, and there are various optimal y.

RO is optimal for constraint-wise uncertainty I

$$
\begin{aligned}
\min_{x, y(\cdot)} \quad & x \\
\text{s.t.} \quad & x - y_1(\xi) \leq \xi_1 && \forall \xi_1 \in [0, \tfrac{1}{2}] \\
& -x + y_1(\xi) + y_2(\xi) \leq \xi_2 + 3\xi_3 + \frac{3}{2} && \forall \xi_2, \xi_3 \in [0,1]^2 \\
& 1 \leq y_1(\xi) \leq \frac{3}{2} \\
& 1 \leq y_2(\xi)
\end{aligned}
$$

➥ this problem has constraint-wise uncertainty because the ξ_1 affect only the first constraint, and the ξ_2, ξ_3 affect only the second constraint.

RO is optimal for constraint-wise uncertainty II

$$\begin{cases} \max\left\{1, 0.5+\xi_1\right\} & \leq & y_1(\xi) & \leq & \min\left\{1.5, 1+\xi_2+3\xi_3\right\} \\ 1 & \leq & y_2(\xi) & \leq & 2+\xi_2+3\xi_3-y_1(\xi) \end{cases}$$

- ➥ ξ_1 appears only on the left-hand side of the first constraint, so we can replace it with its maximum value 0.5;
similarly, ξ_2, ξ_3 appear only on the right-hand side of the second constraint, so we can replace them with their minimum value 0.
- ➥ The optimal recourse is $y_1(\xi) = y_2(\xi) = 1$, that is a constant recourse.

More generally, if the uncertainty is constraint-wise, then the optimal recourse is constant.

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Piecewise-linear decision rules are optimal

For linear two-stage robust optimization problems, we can find an optimal recourse that is piecewise-linear in the uncertainty ξ.

Contents

Two-stage robust linear problem setting

We consider the following two-stage robust linear problem:

$$\begin{aligned}
\min_{x,y(\cdot)} \quad & c^\top x + \max_{\xi \in R} \quad d^\top y_\xi \\
\text{s.t.} \quad & A_\xi x + B_\xi y_\xi \leq b_\xi & \forall \xi \in R \\
& x \in X, y_\xi \geq 0 & \forall \xi \in R
\end{aligned}$$

Which can be reformulated as:

$$\min_{x \in X} c^\top x + V(x) \quad \text{where} \quad V(x) = \max_{\xi \in R} \tilde{V}(x, \xi)$$

With

$$\tilde{V}(x, \xi) := \min_{y_\xi \geq 0} \left\{ d^\top y_\xi \;\middle|\; A_\xi x + B_\xi y_\xi \leq b_\xi \right\}$$

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Bender's decomposition for two-stage problems

- The function $\tilde{V}(\cdot, \xi)$ is polyhedral.
- The function $V(x)$ is convex in x.
- If R is a polyhedron, function $V(x)$ is polyhedral.

➥ We can use a Bender's decomposition approach to solve the problem.

Contents

Upper bound through decision rules

$$\min_{x\in\mathbb{R}^p, y(\cdot):R\mapsto\mathbb{R}^q} \quad \max_{\xi\in R} \quad J(x,\xi,y(\xi))$$
$$\text{s.t.} \quad g(x,\xi,y(\xi)) \leq 0 \qquad \forall \xi \in R$$

An idea is to restrict the recourse decision to be a parametric function (a.k.a decision rule) of the uncertainty ξ,

$$y(\xi) = \pi_\theta(\xi)$$

and see the parameter θ as a first-stage decision variable:

$$\min_{x\in\mathbb{R}^p,\theta} \quad \max_{\xi\in R} \quad J(x,\xi,\pi_\theta(\xi))$$
$$\text{s.t.} \quad g(x,\xi,\pi_\theta(\xi)) \leq 0 \qquad \forall \xi \in R$$

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Two-stage robust linear problem setting

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\begin{aligned}
\min_{x, y(\cdot)} \quad & c^\top x + \max_{\xi \in R} \quad d^\top y_\xi \\
\text{s.t.} \quad & A_\xi x + B_\xi y_\xi \le b_\xi & \forall \xi \in R \\
& x \in \mathbb{R}^p_+, y_\xi \in \mathbb{R}^q_+ & \forall \xi \in R
\end{aligned}
$$

Linear Decision Rules

We consider a linear decision rule:

$$y_\xi = V\xi + u$$

Thus, the two-stage problem becomes:

$$\begin{aligned}
\min_{x,u,V} \quad & c^\top x + \max_{\xi \in R} \quad d^\top (V\xi + u) & \\
\text{s.t.} \quad & A_\xi x + B_\xi V\xi + B_\xi u \leq b_\xi & \forall \xi \in R \\
& x \geq 0,\ V\xi + u \geq 0 & \forall \xi \in R
\end{aligned}$$

As this problem is now one-stage, we can reformulate each constraint independently, using the methods seen previously.

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Linear Decision Rules

- Using LDR, we go from a two-stage problem to a one-stage problem, with increased an number of variables.
- Any LDR, optimal or not, yields a valid upper bound on the optimal value of the two-stage problem.
- For two-stage linear problems, with right-hand side uncertainty only, where R is a simplex, LDR are optimal.
- We can use LDR for multistage problems, non-anticipativity constraints being represented through constraints on the matrix V.
- LDR are not suitable for integer recourse variables.

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Partition methodology

In order to simplify the problem, we can partition the uncertainty set R into a finite number of subsets $R_1, \dots, R_K$, and associate a recourse to each y_k.
Thus, the two-stage problem becomes:

$$\begin{aligned} \min_{x \in \mathbb{R}^p, (y_k)_{k \in [K]}} \quad & \max_{k \in K} \max_{\xi \in R_k} \quad J(x, \xi, y_k) \\ \text{s.t.} \quad & g(x, \xi, y_k \leq 0 \qquad \forall k \in [K], \forall \xi \in R_k \end{aligned}$$

D. Bertsimas, D. Brown, C. Caramanis
Theory and applications of robust optimization
Siam Review, 2011.

D. Bertsimas and D. Den Hertog
Robust and adaptive optimization
Dynamic Ideas, 2022.

BL Gorissen, I. Yanikoglu and D. den Hertog
A practical guide to robust optimization
Omega, 2015.

D. Bertsimas and M. Sim
The price of robustness
Operations research, 2004.

A. Ben Tal, L El Ghaoui, A. Nemirovski
Robust optimization
Springer, 2009.