

Optimization Under Uncertainty - MPRO - 3 hours

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This exam has four independent exercises. Course documents authorized. No communication with other people.

Usefull recalls

i) We have the following result:

$$\forall \xi \sim (\mu, \Sigma), \quad \mathbb{P}(\xi^\top x \leq \alpha) \geq 1 - \varepsilon \quad \Leftrightarrow \quad \mu^\top x \leq \alpha - \sqrt{\frac{1 - \varepsilon}{\varepsilon}} \sqrt{x^\top \Sigma x}, \quad (1)$$

where $\varepsilon \in (0, 1)$, $x \in \mathbb{R}^n$ and $\xi \sim (\mu, \Sigma)$ represent any random vector with mean $\mu \in \mathbb{R}^n$ and covariance matrix Σ .

1 Industrial microgrid energy production (6 points)

We consider an isolated industrial complex with intensive energy consumption. We consider the problem of satisfying the demand load over 24 hours, with 15 minutes time-step, at minimal expected cost, using onsite production and storage. The microgrid has the following elements:

- A known demand to be met d_t (in MW) for each time step;
 - A solar production ξ_t , which is modeled as a stagewise independent finitely supported process. We consider 10 possible production level ξ_t^j , with probability π_t^j . We assume that, at the beginning of stage t , the realization of the production for the next stage is known.
 - A thermal generator, with cost c per MWh, and a maximum production of 2 MW, and a ramp constraint (maximum variation between two consecutive time steps) of 0.5 MW.
 - A battery, with a maximum charge/discharge rate of 1 MW, and a maximum capacity of 5 MWh. Further, the battery has a charging efficiency of 0.9 (i.e., you need to consume 1MWh to charge 0.9MWh in the battery).
 - A flying wheel, which act as a battery. Its capacity is 1 MWh, its maximum charging/discharging rate is 0.4MW. It has a perfect charging/discharging rate, but loose 1% of it's charge between the end of step t and the beginning of $t + 1$.
 - If the demand is exceeded, then energy is lost without additional cost. If demand is not met, there is a penalty of 1000€per MWh.
1. (1 point) Model this problem as a multistage stochastic program. Precise the information structure. If there are missing information, point them out and provide reasonable modeling options.
 2. (0.5 points) Are we in a complete or incomplete recourse setting? Justify.
 3. (2 points) Show that this problem can be solved by Dynamic Programming. Explicit the state, control, noise and dynamic. Write the associated Bellman equation.
 4. (1 point) Estimate the number of operations required to solve this problem by Dynamic Programming.
 5. (0.5 points) Can you use SDDP to solve this problem? Justify.
 6. (2 points) We now assume that the thermal generator can be either off or on with at least 0.5MW. How can it be modeled? What does it change for the resolution of the problem? Assuming that discretized DP is too numerically challenging, propose an approach to provide an upper and lower bound for this new problem.

2 Fast and slow product ordering (5 points)

Consider a company that can order a product whose demand is uncertain by boat in advance (unlimited quantity) at price 1 per unit or by plane at price 4 per unit once the demand is known (maximum of 15 units). Demand has to be met. Unsold product is lost. We aim at minimizing the expected ordering cost.

Demand is assumed to be 10 with probability 0.4, 20 with probability 0.4 and 30 with probability 0.2.

- (0.5 points) Justify that this problem is a two-stage stochastic program by specifying first and second stage variable and write the extensive formulation of this problem.
- (0.5 points) Are we in a complete recourse setting? A relatively complete recourse setting? If not give an explicit first stage constraint yielding relatively complete recourse.
- (0.5 points) Give the anticipative lower bound.
- (0.5 points) Gives the open-loop upper bound.
- (1.5 points) Can we apply the L-Shaped method to this problem? If so give the master problem and slave problems at iteration k (multicut version).
- (1.5 points) Can we apply the Progressive Hedging algorithm to this problem? If so give the master problem and slave problems at iteration k .

3 A robust optimization problem (3 points)

We consider the following optimization problem:

$$\min_{x \in \mathbb{R}_+^3} \quad 3x_1 + 2x_2 + x_3 \quad (2a)$$

$$\text{s.t.} \quad x_1 + \alpha x_2 + \beta x_3 \geq 1 \quad \forall (\alpha, \beta) \in R \quad (2b)$$

$$x_2 \geq 0.5 \quad (2c)$$

$$(2d)$$

where $R = \{(\alpha, \beta) \mid |\alpha| + |\beta| \leq 1\}$.

- (1 point) Give and justify an extensive formulation of problem (2).
- (2 points) Give the pseudo-code of a constraint generation approach, using only a linear program solver, to solve problem (2). Run the algorithm until convergence.

4 A distributionally robust optimization problem (6 points)

We consider the following optimization problem:

$$\min_{x \in \mathbb{R}^n} \quad c^\top x \quad (3a)$$

$$\text{s.t.} \quad \mathbb{P}(\mathbf{a}_i^\top x \leq \alpha_i) \geq 1 - \varepsilon \quad \forall i \in [m] \quad (3b)$$

$$Cx \leq d \quad (3c)$$

where \mathbf{a}_i^\top is the i -th row of the random matrix $A \in \mathbb{R}^{m \times n}$, $\alpha_i \in \mathbb{R}$, $C \in \mathbb{R}^{p \times n}$ and $d \in \mathbb{R}^p$.

- (1 point) Let B be the unit ball of \mathbb{R}^n (i.e., $B = \{\zeta \in \mathbb{R}^n \mid \|\zeta\|_2 \leq 1\}$). In Problem (3) replace constraint (3b) by the robust constraint $\mathbf{a}_i^\top x \leq \alpha_i$, for all $\mathbf{a}_i \in \mu_i + \Delta_i B$, where $\Delta_i \in \mathbb{R}^{n \times n}$. Reformulate the resulting problem as an SOCP.
- (1 point) Assume now that, for $i \in [m]$, we have $\mathbb{P}(\mathbf{a}_i^\top \in \mu_i + \Delta_i B) \geq 1 - \varepsilon$. What is the relation between the SOCP obtained in the previous question and Problem (3)?
- (3 points) We now assume that \mathbf{a}_i^\top is a random vector with mean μ_i and covariance matrix Σ_i . Using the result recalled in i), suggest a distributionally robust counterpart of Problem (3) that admits a tractable reformulation. Give the explicit form of the reformulation, and classify it (LP, SOCP, MILP...). What is the relationship this distributionally robust counterpart and Problem (3)?
- (1 point) If we now add some integrity constraint $x_i \in \mathbb{Z}^i$, for $i \in I$, what is the impact on the previous questions?