Optimization Under Uncertainty - MPRO - 3 hours

Vincent Leclère

22/12/2023

This exam has four independent exercises. Course documents authorized. No communication with other people.

Usefull recalls

i) We have the following result:

$$\forall \boldsymbol{\xi} \sim (\mu, \Sigma), \quad \mathbb{P}(\boldsymbol{\xi}^\top x \leq \alpha) \geq 1 - \varepsilon \quad \Leftrightarrow \quad \mu^\top x \leq \alpha - \sqrt{\frac{1-\varepsilon}{\varepsilon}}\sqrt{x^\top \Sigma x}, \tag{1}$$

where $\varepsilon \in (0,1)$, $x \in \mathbb{R}^n$ and $\boldsymbol{\xi} \sim (\mu, \Sigma)$ represent any random vector with mean $\mu \in \mathbb{R}^n$ and covariance matrix Σ.

1 Industrial microgrid energy production (6 points)

We consider an isolated industrial complex with intensive energy consumption. We consider the problem of satisfying the demand load over 24 hours, with 15 minutes time-step, at minimal expected cost, using onsite production and storage. The microgrid has the following elements:

- A known demand to be met d_t (in MW) for each time step;
- A solar production $\boldsymbol{\xi}_t$, which is modeled as a stagewise independent finitely supported process. We consider 10 possible production level ξ_t^j, with probability π_t^j. We assume that, at the beginning of stage t, the realization of the production for the next stage is known.
- A thermal generator, with cost c per MWh, and a maximum production of 2 MW, and a ramp constraint (maximum variation between two consecutive time steps) of 0.5 MW.
- A battery, with a maximum charge/discharge rate of 1 MW, and a maximum capacity of 5 MWh. Further, the battery has a charging efficiency of 0.9 (i.e., you need to consume 1MWh to charge 0.9MWh in the battery.).
- A flying wheel, which act as a battery. Its capacity is 1 MWh, its maximum charging/discharging rate is 0.4MW. It has a perfect charging/discharging rate, but loose 1% of it's charge between the end of step t and the beginning of $t+1$.
- If the demand is exceeded, then energy is lost without additional cost. If demand is not met, there is a penalty of 1000€per MWh.

1. (1 point) Model this problem as a multistage stochastic program. Precise the information structure. If there are missing information, point them out and provide reasonable modeling options.

Solution:

$$
\begin{aligned}
\min \quad & \mathbb{E}\Big[\sum_{t=1}^{96} c\boldsymbol{u_t^{th}} + 1000/4\boldsymbol{\eta}_t\Big] & \\
\text{s.t.} \quad & \boldsymbol{u_t^{th}} + \boldsymbol{u_t^{bat}} + \boldsymbol{u_t^{fw}} + \boldsymbol{\xi_t} + \boldsymbol{\eta_t} \geq d_t & \forall t \in [96] \\
& \boldsymbol{s_t^{bat}} = \boldsymbol{s_{t-1}^{bat}} - (\boldsymbol{u_t^{bat}})^+ + 0.9(\boldsymbol{u_t^{bat}})^- & \forall t \in [96] \\
& \boldsymbol{s_t^{fw}} = 0.99(\boldsymbol{s_{t-1}^{fw}} - \boldsymbol{u_t^{fw}}) & \forall t \in [96] \\
& 0 \leq \boldsymbol{u_t^{th}} \leq 2/4,\ |\boldsymbol{u_t^{bat}}| \leq 1/4,\ |\boldsymbol{u_t^{fw}}| \leq 0.4/4,\ \boldsymbol{\eta}_t \geq 0 & \forall t \in [96] \\
& |\boldsymbol{u_t^{th}} - \boldsymbol{u_{t-1}^{th}}| \leq 0.5/4 & \forall t \in [96] \\
& 0 \leq \boldsymbol{s_t^{bat}} \leq 5,\ 0 \leq \boldsymbol{s_t^{fw}} \leq 1 & \forall t \in [96] \\
& \boldsymbol{u}_t, \boldsymbol{\eta}_t \preceq \boldsymbol{\xi}_{[t]} & \forall t \in [96] \\
& s_0^{bat} = 0,\ s_0^{fw} = 0,\ u_0^{th} = 0. &
\end{aligned}
$$

2. (0.5 points) Are we in a complete or incomplete recourse setting? Justify.

Solution: We are not in complete recourse as we cannot get back to an admissible battery level if we start too high.

Relatively complete recourse: from any admissible state, we can stay in an admissible state with $u_t = 0$ and $\eta_t = d_t - \xi_t$.

3. (2 points) Show that this problem can be solved by Dynamic Programming. Explicit the state, control, noise and dynamic. Write the associated Bellman equation.

Solution: Noises are stagewise independent.

State: $s_t^{bat}, s_t^{fw}, u_{t-1}^{th}$, control: $u_t^{bat}, u_t^{fw}, \eta_t$, noise: ξ_t.

$$
\begin{aligned}
\dot{V}_t(s_{t-1}^{bat}, s_{t-1}^{fw}, u_{t-1}^{th}; \xi_t) = \min_{u_t^{th}, u_t^{fw}, u_t^{bat}, \eta_t} \quad & cu_t^{th} + 250\eta_t + V_{t+1}(s_t^{bat}, s_t^{fw}, u_t^{th}) \\
\text{s.t.} \quad & u_t^{th} + u_t^{bat} + u_t^{fw} + \xi_t + \eta_t \geq d_t \\
& s_t^{bat} = s_{t-1}^{bat} - (u_t^{bat})^+ + 0.9(u_t^{bat})^- \\
& s_t^{fw} = 0.99(s_{t-1}^{fw} - u_t^{fw}) \\
& 0 \leq u_t^{th} \leq 2/4,\ |u_t^{bat}| \leq 1/4,\ |u_t^{fw}| \leq 0.4/4,\ \eta_t \geq 0 \\
& |u_t^{th} - u_{t-1}^{th}| \leq 0.5/4 \\
& 0 \leq s_t^{bat} \leq 5,\ 0 \leq s_t^{fw} \leq 1
\end{aligned}
$$

$$
V_t(s_{t-1}^{bat}, s_{t-1}^{fw}, u_{t-1}^{th}) = \sum_{j=1}^{10} \pi_t^j \dot{V}_t(s_{t-1}^{bat}, s_{t-1}^{fw}, u_{t-1}^{th}; \xi_t^j)
$$

4. (1 point) Estimate the number of operations required to solve this problem by Dynamic Programming.

Solution: With a 0.1MW discretization, we have 96 time-step; state: 50 battery level, 10 flying wheel level, 5 thermal generator level; control: 5 battery level, 3 flying wheel level, 3 thermal generator level; noise: 10 solar production level. Thus we have $96 \times 50 \times 10 \times 5 \times 5 \times 3 \times 3 \times 10 \approx 10^8$ operations.

5. (0.5 points) Can you use SDDP to solve this problem? Justify.

Solution: Yes, we have a linear problem with relatively complete recourse.

6. (2 points) We now assume that the thermal generator can be either off or on with at least 0.5MW. How can it be modeled? What does it change for the resolution of the problem? Assuming that discretized DP is too numerically challenging, propose an approach to provide an upper and lower bound for this new problem.

Solution: We need to add a binary variable z_t and a constraint $0.125z_t \leq u_t^{th} \leq 0.5z_t$.

We can still use DP, with roughly the same number of operations.

We can use SDDP to provide a lower bound (the same as before - were integrity constraints are relaxed). We can obtained an upper bound by simulating the policy induced by the SDDP algorithm: at each iteration solve the one-stage DP problem with binary variables, using the SDDP-computed cost-to-go function. This can be simulated with a Monte-Carlo approach.

2 Fast and slow product ordering (5 points)

Consider a company that can order a product whose demand is uncertain by boat in advance (unlimited quantity) at price 1 per unit or by plane at price 4 per unit once the demand is known (maximum of 15 units). Demand has to be met. Unsold product is lost. We aim at minimizing the expected ordering cost.

Demand is assumed to be 10 with probability 0.4, 20 with probability 0.4 and 30 with probability 0.2.

1. (0.5 points) Justify that this problem is a two-stage stochastic programm by specifying first and second stage variable and write the extensive formulation of this problem.

Solution: First stage variable: x (number of units ordered by boat), second stage variable: y (number of units ordered by plane).

$$
\begin{aligned}
\min \quad & x + 4(0.4y_1 + 0.4y_2 + 0.2y_3) & \\
\text{s.t.} \quad & x + y_1 \geq 10 & \\
& x + y_2 \geq 20 & \\
& x + y_3 \geq 30 & \\
& x, y_i \geq 0 & \forall i \in [3] \\
& y_i \leq 15 & \forall i \in [3]
\end{aligned}
$$

2. (0.5 points) Are we in a complete recourse setting? A relatively complete recourse setting? If not give an explicit first stage constraint yielding relatively complete recourse.

Solution: We are not in a relatively complete recourse setting if we do not add the constraint $x \geq 15$ in the first stage.

3. (0.5 points) Give the anticipative lower bound.

Solution: In the anticipative framework we order everything by boat, yielding a cost of $0.4 \times 10 + 0.4 \times 20 + 0.2 \times 30 = 18$.

4. (0.5 points) Gives the open-loop upper bound.

Solution: In the open-loop framework we order everything by boat, yielding a cost of $1 \times 30 = 30$.

5. (1.5 points) Can we apply the L-Shaped method to this problem? If so gives the master problem and slave problems at iteration k (multicut version).

Solution: We can apply the L-Shaped method to this linear stochastic problem. The master problem reads

$$
\begin{aligned}
\min \quad & x + (0.4\theta_1 + 0.4\theta_2 + 0.2\theta_3) & \\
\text{s.t.} \quad & x \geq 15 & \\
& \theta_s \geq \alpha_s^\kappa (x - x^\kappa) + \beta_s^\kappa & \forall s \in [3], \forall \kappa \in [k]
\end{aligned}
$$

The slave problem are, for $d_s \in \{10, 20, 30\}$,

$$\begin{aligned} \beta_s^{k+1} = \min_{y \geq 0} \quad & 4y \\ \text{s.t.} \quad & y \geq d_s - x \\ & x = x^k & [\alpha_s^{k+1}] \end{aligned}$$

6. (1.5 points) Can we apply the Progressive Hedging algorithm to this problem? If so give the master problem and slave problems at iteration k.

3 A robust optimization problem (3 points)

We consider the following optimization problem:

$$\begin{aligned} \min_{x \in \mathbb{R}^3_+} \quad & 3x_1 + 2x_2 + x_3 & & \text{(2a)} \\ \text{s.t.} \quad & x_1 + \alpha x_2 + \beta x_3 \geq 1 & \forall (\alpha, \beta) \in R \quad & \text{(2b)} \\ & x_2 \geq 0.5 & & \text{(2c)} \\ & & & \text{(2d)} \end{aligned}$$

where $R = \{(\alpha, \beta) \mid |\alpha| + |\beta| \leq 1\}$.

1. (1 point) Give and justify an extensive formulation of problem (2).

Solution: As R is a polytope, an the cost is linear, we can replace R by its vertices, yielding the following extensive formulation:

$$\begin{aligned} \min_{x \in \mathbb{R}^3_+} \quad & 3x_1 + 2x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 \geq 1 \\ & x_1 + x_2 - x_3 \geq 1 \\ & x_1 - x_2 + x_3 \geq 1 \\ & x_1 - x_2 - x_3 \geq 1 \\ & x_2 \geq 0.5 \end{aligned}$$

2. (2 points) Give the pseudo-code of a constraint generation approach, using only a linear program solver, to solve problem (2). Run the algorithm until convergence.

Solution:

1. Set $K = 1$, $(\alpha^1, \beta^1) = (0, 0)$.
2. Solve the following linear program:

$$\begin{aligned} \min_{x \in \mathbb{R}^3_+} \quad & 3x_1 + 2x_2 + x_3 \\ \text{s.t.} \quad & x_1 + \alpha^k x_2 + \beta^k x_3 \geq 1 & \forall k \in [K] \\ & x_2 \geq 0.5 \end{aligned}$$

with x^k the optimal solution.

3. Solve the following linear program:

$$\begin{aligned} \min_{\alpha,\beta} \quad & x_1^k + x_2^k\alpha + x_3^k\beta \\ \text{s.t.} \quad & |\alpha| + |\beta| \leq 1 \end{aligned}$$

with $(\alpha^{K+1}, \beta^{K+1})$ the optimal solution, and z^{K+1} the optimal value.

4. If $z^{K+1} \geq 1$, stop. Otherwise, set $K = K + 1$ and go to step 2.

Here we have $x^1 = (1, 0.5, 0)$, $z^1 = 0.5$, $(\alpha^2, \beta^2) = (-1, 0)$, $x^2 = (1.5, 0.5, 0)$, $z^2 = 1$, stop.

4 A distributionally robust optimization problem (6 points)

We consider the following optimization problem:

$$\min_{x \in \mathbb{R}^n} \quad c^\top x \tag{3a}$$

$$\text{s.t.} \quad \mathbb{P}(\boldsymbol{a}_i^\top x \leq \alpha_i) \geq 1 - \varepsilon \qquad \forall i \in [m] \tag{3b}$$

$$Cx \leq d \tag{3c}$$

where $\boldsymbol{a}_i^\top$ is the i-th row of the random matrix $A \in \mathbb{R}^{m \times n}$, $\alpha_i \in \mathbb{R}$, $C \in \mathbb{R}^{p \times n}$ and $d \in \mathbb{R}^p$.

1. (1 point) Let B be the unit ball of $\mathbb{R}^n$ (i.e., $B = \{\zeta \in \mathbb{R}^n \mid \|\zeta\|_2 \leq 1\}$). In Problem (3) replace constraint (3b) by the robust constraint $a_i^\top x \leq \alpha_i$, for all $a_i \in \mu_i + \Delta_i B$, where $\Delta_i \in \mathbb{R}^{n \times n}$. Reformulate the resulting problem as an SOCP.

Solution: We have

$$\begin{aligned} \forall a_i \in \mu_i + \Delta_i B, \quad a_i^\top x \leq \alpha_i \Leftrightarrow & \sup_{a_i \in \mu_i + \Delta_i B} a_i^\top x \leq \alpha_i \\ \Leftrightarrow & \mu_i^\top x + \sup_{\|\zeta\|_2 \leq 1} (\Delta_i \zeta)^\top x \leq \alpha_i \\ \Leftrightarrow & \mu_i^\top x + \|\Delta_i^\top x\|_2 \leq \alpha_i \end{aligned}$$

Thus the robust problem reads

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & \mu_i^\top x + \|\Delta_i^\top x\|_2 \leq \alpha_i & \forall i \in [m] \\ & Cx \leq d \end{aligned}$$

which is an SOCP.

2. (1 point) Assume now that, for $i \in [m]$, we have $\mathbb{P}(\boldsymbol{a}_i^\top \in \mu_i + \Delta_i B) \geq 1 - \varepsilon$. What is the relation between the SOCP obtained in the previous question and Problem (3)?

Solution: A solution of the above SOCP is a feasible solution of Problem (3), and it's value is an upper bound of the optimal value of Problem (3).

3. (3 points) We now assume that $\boldsymbol{a}_i^\top$ is a random vector with mean μ_i and covariance matrix Σ_i. Using the result recalled in i), suggest a distributionally robust counterpart of Problem (3) that admits a tractable reformulation. Give the explicit form of the reformulation, and classify it (LP, SOCP, MILP...). What is the relationship this distributionally robust counterpart and Problem (3)?

Solution: We can consider the following distributionally robust counterpart:

$$\begin{aligned}
\min_{x\in\mathbb{R}^n} \quad & c^\top x \\
\text{s.t.} \quad & \sup_{\boldsymbol{a}_i\sim(\mu_i,\Sigma_i)} \mathbb{P}(\boldsymbol{a}_i^\top x \leq \alpha_i) \geq 1-\varepsilon & \forall i \in [m] \\
& Cx \leq d
\end{aligned}$$

Again, a solution of this problem is feasible for Problem (3), and it's value is an upper bound of the optimal value of Problem (3).

Further, using the result recalled in i), we can reformulate this problem as an SOCP:

$$\begin{aligned}
\min_{x\in\mathbb{R}^n} \quad & c^\top x \\
\text{s.t.} \quad & \mu_i^\top x + \sqrt{\frac{1-\varepsilon}{\varepsilon}}\|\Sigma_i^{1/2}x\| \leq \alpha_i & \forall i \in [m] \\
& Cx \leq d
\end{aligned}$$

4. (1 point) If we now add some integrity constraint $x_i \in \mathbb{Z}^i$, for $i \in I$, what is the impact on the previous questions?

Solution: The integrity does not change the previous answers. The reformulations are now MISOCPs.