# Optimality conditions

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# Why should I bother to learn this stuff?

- Optimality conditions enable to solve exactly some easy optimization problems (e.g. in microeconomics, some mechanical problems...)
- Optimality conditions are used to derive algorithms for complex problem
- $\Longrightarrow$ fundamental both for studying optimization as well as other science

# Contents


1. Optimization problem [BV 4.1]
2. Unconstrained case [BV 4.2]
3. First order optimality conditions [B.V 5.5]
4. Wrap-up

# Optimization problem: vocabulary ♡

Generically speaking, an optimization problem is

$$\underset{x \in X}{\text{Min}} \qquad f(x) \qquad (P)$$

where

- $f : \mathbb{R}^n \to \mathbb{R}$ is the objective function (a.k.a. cost function),
- $X$ is the feasible set,
- $x \in X$ is an admissible decision variables or a solution,
- $x^\sharp \in X$ such that $val(P) = f(x^\sharp) = \inf_{x \in X} f(x)$ is an optimal solution,
- if $X = \mathbb{R}^n$ the problem is unconstrained,
- if $X$ and $f$ are convex, then the problem is convex,
- if $X$ is a polyhedron and $f$ linear then the problem is linear,
- if $X$ is a convex cone and $f$ linear then the problem is conic.

# Optimization problem: explicit formulation ♡

The previous optimization problem is often defined explicitly in the following standard form

$$\begin{aligned}
\underset{x\in\mathbb{R}^n}{\text{Min}} \quad & f(x) && (P)\\
\text{s.t.} \quad & g_i(x) = 0 && \forall i \in [n_E]\\
& h_j(x) \leq 0 && \forall j \in [n_I]
\end{aligned}$$

with

$$X := \big\{x \in \mathbb{R}^n \mid \forall i \in [n_E], \quad g_i(x) = 0, \quad \forall j \in [n_I], \quad h_j(x) \leq 0\big\}.$$

- $(P)$ is a differentiable optimization problem if $f$ and $\{g_i\}_{i\in[n_E]}$ and $\{h_j\}_{j\in[n_I]}$ are differentiable.
- $(P)$ is a convex differentiable optimization problem if $f$, and $h_j$ (for $j \in [n_I]$) are convex differentiable and $g_i$ (for $i \in [n_E]$) are affine.
  ♣ Exercise: Show that in this case $X$ is convex.

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# A few remarks and tricks

- We can always write an abstract optimization problem in standard form (exercise!)
- For a given optimization problem there is an infinite number of possible standard forms (exercise!)
- We can always find an equivalent problem in dimension $\mathbb{R}^{n+1}$ with linear cost (exercise!)
- A minimization problem with $X = \emptyset$ has value $+\infty$ (by convention)
- A minimization problem has value $-\infty$ iff there exists a sequence $x_n \in X$ such that $f(x_n) \to -\infty$
- Maximizing $f$ is just minimizing $-f$

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# Differentiable case

♡

**Theorem**

*Assume that $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is differentiable at $x^\sharp$.*

1. *If $x^\sharp$ is an unconstrained local minimizer of $f$ then $\nabla f(x^\sharp) = 0$.*
2. *If in addition $f$ is convex, then $\nabla f(x^\sharp) = 0$ iff $x^\sharp$ is a global minimizer.*

Proof:

1. Assume $\nabla f(x^\sharp) \neq 0$. DL of order 1 at $x^\sharp$ show that $f(x^\sharp - t\nabla f(x^\sharp)) < f(x^\sharp)$ for $t > 0$ small enough.
2. $f(y) \geq f(x^\sharp) + \langle \nabla f(x^\sharp), y - x^\sharp \rangle$.

# Differentiable case ♡

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*Assume that* $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ *is differentiable at* $x^\sharp$.

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**Theorem**

*Consider* $f : \mathbb{R}^n \to \bar{\mathbb{R}}$. *Then* $x^\sharp$ *is a global minimum iff*

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**Theorem**

*Consider a proper convex function* $f : \mathbb{R}^n \to \bar{\mathbb{R}}$*, and* $X$ *a closed convex set, such that* $\mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{ri}(X) \neq \emptyset$*.*
*Then* $x^\sharp$ *is a minimizer of* $f$ *on* $X$ *iff there exists* $g \in \partial f(x^\sharp)$ *such that* $-g \in N_X(x^\sharp)$*.*

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*Then $x^\sharp$ is a minimizer of $f$ on $X$ iff there exists $g \in \partial f(x^\sharp)$ such that $-g \in N_X(x^\sharp)$.*

proof : The technical assumption ensures that $\partial(f + \mathbb{I}_X) = \partial f + \partial(\mathbb{I}_X)$.
As $\partial(\mathbb{I}_X) = N_X$, we have, $0 \in \partial(f + \mathbb{I}_X)(x^\sharp)$ iff there exists $g \in \partial f(x^\sharp)$ such that $-g \in N_X(x^\sharp)$.

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# Tangent cones

For $f : \mathbb{R}^n \to \mathbb{R}$, we consider an optimization problem of the form

$$\underset{x \in X}{\text{Min}} \qquad f(x).$$

## Definition

We say that $d \in \mathbb{R}^n$ is tangent to $X$ at $x \in X$ if there exists a sequence $x_k \in X$ converging to $x$ and a sequence $t_k \searrow 0$ such that

$$d = \lim_k \frac{x_k - x}{t_k}.$$

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Let $T_X(x)$ be the tangent cone of $X$ at $x$, that is, the set of all tangent to $X$ at $x$.

Equivalently,

$$T_X(x) = \{ \, d \in \mathbb{R}^n \ \mid \exists t_k \searrow 0, \ \exists d_k \to d, \ x + t_k d_k \in X \}$$

# Optimality conditions - differentiable case

Consider a function $f : \mathbb{R}^n \to \mathbb{R}$ and the optimization problem

$$(P) \qquad \underset{x \in X}{\text{Min}} \qquad f(x).$$

If $x^\sharp \notin \text{int}(X)$ we do not necessarily need to have $\nabla f(x^\sharp) = 0$, indeed we just to have $\langle d, \nabla f(x^\sharp) \rangle \leq 0$ for all "admissible" direction $d$.

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## Theorem

*Assume that $f$ is differentiable at $x^\sharp$.*

1. *If $x^\sharp$ is a local minimizer of $(P)$ we have*

$$\nabla f(x^\sharp) \in \left[T_X(x^\sharp)\right]^\oplus. \qquad (*)$$

2. *If $f$ and $X$ are both convex, and $(*)$ holds, then $x^\sharp$ is an optimal solution of $(P)$*

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2. *If $f$ and $X$ are both convex, and $(*)$ holds, then $x^\sharp$ is an optimal solution of $(P)$*

♠ Exercise: Prove this result.

# Convex case

Let $K_X^{ad}(x)$ be the cone of admissible direction

$$K_X^{ad}(x) := \{ t(y-x) \in \mathbb{R}^n \quad | \quad y \in X, \quad t \geq 0 \}$$

**Lemma**

*If $X \subset \mathbb{R}^n$ is convex, and $x \in X$, we have*

$$T_X(x) = \overline{K_X^{ad}(x)}.$$

Recall that

$$T_X(x) = \{ \ d \in \mathbb{R}^n \ \mid \exists t_k \searrow 0, \ \exists d_k \to d, \ x + t_k d_k \in X \}$$

♠ Exercise: Prove this lemma

# Differentiable constraints

We consider the following set of admissible solution

$$X = \Big\{ x \in \mathbb{R}^n \quad | \quad g_i(x) = 0,\ i \in [n_E] \quad h_j(x) \leq 0,\ j \in [n_j] \Big\},$$

where $g$ and $h$ are differentiable functions.

Recall that the tangent cone is given by

$$T_X(x) = \{\ d \in \mathbb{R}^n \ | \ \exists t_k \searrow 0,\ \exists d_k \to d,\ g(x + t_k d_k) = 0,\ h(x + t_k d_k) \leq 0 \}$$

We define the linearized tangent cone

$$\begin{aligned} T_X^\ell(x) := \{\ d \in \mathbb{R}^n \ | & \left\langle \nabla g_i(x)\,, d \right\rangle = 0,\ \forall i \in [n_E] \\ & \left\langle \nabla h_j(x)\,, d \right\rangle \leq 0,\ \forall j \in I_0(x) \} \end{aligned}$$

where

$$I_0(x) := \Big\{ j \in [n_I] \quad | \quad h_j(x) = 0 \Big\}.$$

# Constraint qualifications ♢

We always have

$$T_X(x) \subset T_X^\ell(x).$$

♣ Exercise: Prove it.

We say that the constraints are qualified at $x$ if

$$T_X(x) = T_X^\ell(x).$$

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# Sufficient qualification conditions ♡

Recall that $g$ and $h$ are assumed differentiable.
We denote the index set of active constraints at $x$

$$I_0(x) := \{i \in [n_I] \quad | \quad h_i(x) = 0\}.$$

The following conditions are sufficient qualification conditions at $x$:

1. $g$ and $h_i$ for $i \in I_0(x)$ are locally affine;
2. (Slater) $g$ is affine, $h_j$ are convex, and there exists $x_S$ such that $g(x_S) = 0$ and $h_j(x_S) < 0$ ;
3. (Mangasarian-Fromowitz) For all $\alpha \in \mathbb{R}^{n_E}$ and $\beta \in \mathbb{R}_+^{n_I}$,

$$\sum_{i \in [n_E]} \alpha_i \nabla g_i(x) + \sum_{j \in I_0(x)} \beta_j \nabla h_j(x) = 0 \qquad \Longrightarrow \qquad \alpha = 0 \text{ and } \beta = 0$$

# Expliciting the optimality condition I $\diamondsuit$

Under constraint qualification, the optimality condition reads

$$\nabla f(x) \in \left[T_X^\ell(x)\right]^\oplus$$

where

$$T_X^\ell(x) = \{\, d \in \mathbb{R}^n \mid \underbrace{\left\langle \nabla g_i(x)\,, d\right\rangle = 0, i \in [n_I] \quad \left\langle \nabla h_j(x)\,, d\right\rangle \leq 0, j \in I_0(x)}_{= A_x d \in C} \}.$$

with $A_x = \begin{pmatrix} ((\nabla g_i(x))^\top)_{i \in [n_I]} \\ ((\nabla h_j(x))^\top)_{j \in I_0(x)} \end{pmatrix}$ and $C = \{0\}^{n_E} \times (\mathbb{R}_-)^{n_I}$.

♣ Exercise: Show that $C^\oplus = \mathbb{R}^{n_E} \times (\mathbb{R}_-)^{n_I}$

# Expliciting the optimality condition



Recall that the positive dual cone of a set $K$ is

$$K^{\oplus} := \{ d \in \mathbb{R}^n \quad | \quad \langle d, x \rangle \geq 0, \forall x \in K \}.$$

Let $C$ be a closed convex set. Consider

$$K = A^{-1}C := \{ x \in \mathbb{R}^n \mid Ax \in C \},$$

then

$$K^{\oplus} = \{ A^{\top}\lambda \mid \lambda \in C^{\oplus} \}.$$

♣ Exercise: prove it.

Hence,

$$\nabla f(x) \in \big[ \underbrace{T_X^{\ell}(x)}_{A_x^{-1}C} \big]^{\oplus}$$

$$\iff \quad \exists \lambda \in C^{\oplus}, \quad \nabla f(x) = A_x^{\top}\lambda$$

$$\iff \quad \exists \lambda \in \mathbb{R}^{n_E}, \ \exists \mu \in \mathbb{R}_+^{I_0(x)} \quad \nabla f(x) + \sum_{i=1}^{n_E} \lambda_i \nabla g_i(x) + \sum_{j \in I_0(x)} \mu_j \nabla h_j(x) = 0.$$

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$$\iff \quad \exists \lambda \in \mathbb{R}^{n_E},\ \exists \mu \in \mathbb{R}_+^{I_0(x)} \quad \nabla f(x) + \sum_{i=1}^{n_E} \lambda_i \nabla g_i(x) + \sum_{j \in I_0(x)} \mu_j \nabla h_j(x) = 0.$$

# Karush Kuhn Tucker condition ♡

## Theorem (KKT)

*Assume that the objective function $f$ and the constraint function $g_i$ and $h_j$ are differentiable. Assume that the constraints are qualified at $x$.*

*Then if $x$ is a local minimum of*

$$\min_{\tilde{x}\in\mathbb{R}^n} \Big\{ f(\tilde{x}) \quad | \quad g_i(\tilde{x}) = 0,\ \forall i \in [n_E] \quad h_j(\tilde{x}) \leq 0,\ \forall j \in [n_I] \Big\}$$

*then there exists dual variables $\lambda, \mu$ such that*

$$\begin{cases} \nabla f(x) + \sum_{i=1}^{n_E} \lambda_i \nabla g_i(x) + \sum_{j=1}^{n_I} \mu_j \nabla h_j(x) = 0 & \nabla_x \mathcal{L} = 0 \\ g(x) = 0, \quad h(x) \leq 0 & \textit{Primal feasibility} \\ \lambda \in \mathbb{R}^{n_E}, \quad \mu \in \mathbb{R}_+^{n_I} & \textit{dual feasibility} \\ \mu_j h_j(x) = 0 \qquad \forall j \in [n_I] & \textit{complementarity constraint} \end{cases}$$

# Exercise

Solve the following optimization problem

$$\begin{aligned} \underset{x,y\in\mathbb{R}^2}{\text{Min}} \quad & (x-1)^2 + (y-2)^2 \\ & x \leq y \\ & x + 2y \leq 2 \end{aligned}$$

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# What you have to know

- Basic vocabulary: objective, constraint, admissible solution, differentiable optimization problem
- First order necessary KKT conditions

# What you really should know

- What is a tangent cone
- Sufficient qualification conditions (linear and Slater's)
- That KKT conditions are sufficient in the convex case

# What you have to be able to do

- Write the KKT condition for a given explicit problem and use them to solve said problem

# What you should be able to do

- Check that constraints are qualified