Convexity

V. Leclère (ENPC)

March, 15th 2024

Why should I bother to learn this stuff?

- Convex vocabulary and results are needed throughout the course, especially to obtain optimality conditions and duality relations.
- Convex analysis tools like Fenchel transform appears in modern machine learning theory
- $\Longrightarrow$ fundamental for M2 in continuous optimization
- $\Longrightarrow$ usefull for M2 in operation research, machine learning (and some part of probability or mechanics)

Contents

Affine sets

♡

Let X be a normed vector space (usually $X = \mathbb{R}^n$), and $C \subset X$

- C is affine if it contains any lines going through two distinct points of C, *i.e.*,

$$\forall x, y \in C, \quad \forall \theta \in \mathbb{R}, \qquad \theta x + (1-\theta)y \in C.$$

- The affine hull of C is the set of affine combination of elements of C,

$$\mathrm{aff}(C) := \Big\{ \sum_{i=1}^{K} \theta_i x_i \;\Big|\; \forall x_i \in C,\ \forall \theta_i \in \mathbb{R},\ \sum_{i=1}^{K} \theta_i = 1,\ \forall i \in [K], \forall K \in \mathbb{N} \Big\}$$

- $\mathrm{aff}(C)$ is the smallest affine space containing C.
- The affine dimension of C is the dimension of $\mathrm{aff}(C)$ (*i.e.*,the dimension of the vector space $\mathrm{aff}(C) - x_0$ for $x_0 \in C$).
- The relative interior of C is defined as

$$\mathrm{ri}(C) := \Big\{ x \in C \;\Big|\; \exists r > 0, \quad B(x,r) \cap \mathrm{aff}(C) \subset C \Big\}$$

Affine sets

♡

Let X be a normed vector space (usually $X = \mathbb{R}^n$), and $C \subset X$

- C is affine if it contains any lines going through two distinct points of C, *i.e.*,
$$\forall x, y \in C, \quad \forall \theta \in \mathbb{R}, \qquad \theta x + (1-\theta) y \in C.$$
- The affine hull of C is the set of affine combination of elements of C,
$$\operatorname{aff}(C) := \Big\{ \sum_{i=1}^K \theta_i x_i \;\Big|\; \forall x_i \in C,\ \forall \theta_i \in \mathbb{R},\ \sum_{i=1}^K \theta_i = 1,\ \forall i \in [K], \forall K \in \mathbb{N} \Big\}$$
- $\operatorname{aff}(C)$ is the smallest affine space containing C.
- The affine dimension of C is the dimension of $\operatorname{aff}(C)$ (*i.e.*,the dimension of the vector space $\operatorname{aff}(C) - x_0$ for $x_0 \in C$).
- The relative interior of C is defined as
$$\operatorname{ri}(C) := \Big\{ x \in C \quad \Big| \quad \exists r > 0, \quad B(x,r) \cap \operatorname{aff}(C) \subset C \Big\}$$

Convex sets

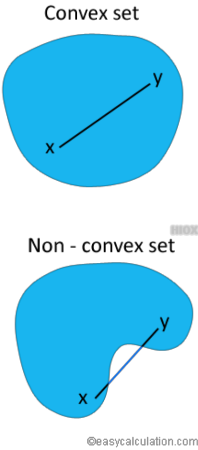
- C is convex if for any two points x and y in C the segment $[x, y] \subset C$, *i.e.*,

$$\forall x, y \in C,\ \forall \theta \in [0, 1],\ \theta x + (1-\theta) y \in C.$$

- The convex hull of C as the set of convex combination of elements of C, *i.e.*,

$$\mathrm{conv}(C) := \Big\{ \sum_{i=1}^{K} \theta_i x_i \mid \ \forall x_i \in C, \ \forall \theta_i \in [0,1],\ \sum_{i=1}^{K} \theta_i = 1,\ \forall i \in [K],\ \forall K \in \mathbb{N} \Big\}$$

- $\mathrm{conv}(C)$ is the smallest convex set containing C.



Cones

- C is a cone if for all $x \in C$ the ray $\mathbb{R}_+ x \subset C$, *i.e.*,

$$\forall x \in C, \quad \forall \theta \in \mathbb{R}_+, \qquad \theta x \in C.$$

- The (convex) conic hull of C is the set of all (convex) conic combination of elements of C *i.e.*,

$$\mathrm{cone}(C) := \Big\{ \sum_{i=1}^{K} \theta_i x_i \mid \forall x_i \in C, \forall \theta_i \in \mathbb{R}_+, \forall i \in [K], \forall K \in \mathbb{N} \Big\}$$

- $\mathrm{cone}(C)$ is the smallest convex cone containing C.
- A cone C is pointed if it does not contain any full line $\mathbb{R}x$ for $x \neq 0$.
- For C convex, $\mathrm{cone}(C) = \bigcup_{t>0} tC$

Examples

Let $X = \mathbb{R}^n$.

- Any affine space is convex.
- Any hyperplane of X can be defined as $H := \{x \in X \mid \ a^\top x = b\}$ for well choosen $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ and is an affine space of dimension $n - 1$.
- H divide X into two half-spaces $\{x \in \mathbb{R}^n \mid a^\top x \leq b$ and $\{x \in \mathbb{R}^n \mid a^\top x \geq b\}$ which are (closed) convex sets.
- For any norm $\| \cdot \|$ the ball $B_{\|\cdot\|}(x_0, r) := \{x \in X \mid \|x - x_0\| \leq r\}$ is a (closed) convex set.
  ♣ Exercise: Prove it.
- The set $C = \{(x, t) \in X \times \mathbb{R} \mid \|x\| \leq t \ \}$ is a cone.
- The set $C = \{x \in X \mid Ax \leq b\}$ where A and b are given is a (closed) convex set called polyhedron.

Examples

Let $X = \mathbb{R}^n$.

- Any affine space is convex.
- Any hyperplane of X can be defined as $H := \{x \in X \mid \ a^\top x = b\}$ for well choosen $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ and is an affine space of dimension $n-1$.
- H divide X into two half-spaces $\{x \in \mathbb{R}^n \mid a^\top x \leq b$ and $\{x \in \mathbb{R}^n \mid a^\top x \geq b\}$ which are (closed) convex sets.
- For any norm $\| \cdot \|$ the ball $B_{\|\cdot\|}(x_0, r) := \{x \in X \mid \|x - x_0\| \leq r\}$ is a (closed) convex set.
  ♣ Exercise: Prove it.
- The set $C = \{(x,t) \in X \times \mathbb{R} \mid \|x\| \leq t \ \}$ is a cone.
- The set $C = \{x \in X \mid Ax \leq b\}$ where A and b are given is a (closed) convex set called polyhedron.

Operations preserving convexity ♡

Assume that all sets denoted by C (indexed or not) are convex.

- $C_1 + C_2$ and $C_1 \times C_2$ are convex sets.
- For any arbitrary index set $\mathcal{I}$ the intersection $\bigcap_{i \in \mathcal{I}} C_i$ is convex.
- Let f be an affine function. Then $f(C)$ and $f^{-1}(C)$ are convex.
- In particular, $C + x_0$, and tC are convex. The projection of C on any affine space is convex.
- The closure $\mathrm{cl}(C)$ and relative interior $\mathrm{ri}(C)$ are convex.

♣ Exercise: Prove these results.

Perspective and linear-fractional function

Let $P : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be the perspective function defined as $P(x, t) = x/t$, with $\mathrm{dom}(P) = \mathbb{R}^n \times \mathbb{R}_+^*$.

Theorem

If $C \subset \mathrm{dom}(P)$ is convex, then $P(C)$ is convex.
If $C \subset \mathbb{R}^n$ is convex, then $P^{-1}(C)$ is convex.

♠ Exercise: Prove this result.

Perspective and linear-fractional function ♢

Let $P : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be the perspective function defined as $P(x,t) = x/t$, with $\mathrm{dom}(P) = \mathbb{R}^n \times \mathbb{R}_+^*$.

Theorem

If $C \subset \mathrm{dom}(P)$ is convex, then $P(C)$ is convex.
If $C \subset \mathbb{R}^n$ is convex, then $P^{-1}(C)$ is convex.

♠ Exercise: Prove this result.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear-fractional function of the form $f(x) := (Ax+b)/(c^\top x + d)$, with $\mathrm{dom}(f) = \{x \,|\, c^\top x + d > 0\}$.

Theorem

If $C \subset \mathrm{dom}(f)$ is convex, then $f(C)$ and $f^{-1}(C)$ are convex.

♣ Exercise: prove this result.

Cone ordering

Let $K \subset \mathbb{R}^n$ be a closed, convex, pointed cone with non-empty interior. We define the cone ordering according to K by

$$x \preceq_K y \quad \iff \quad y - x \in K.$$

♣ Exercise: Prove that $\preceq_K$ is a partial order (*i.e.*,reflexive, antisymmetric, transitive) compatible with scalar product, addition and limits.

Contents

Separation

Let X be a Banach space, and X^* its topological dual (i.e. the set of all continuous linear forms on X).

Theorem (Simple separation)

Let A and B be convex non-empty, disjunct subsets of X. There exists a separating hyperplane $(x^*, \alpha) \in X^* \times \mathbb{R}$ such that

$$\langle x^*, a \rangle \leq \alpha \leq \langle x^*, b \rangle \qquad \forall a, b \in A \times B.$$

Theorem (Strong separation)

Let A and B be convex non-empty, disjunct subsets of X. Assume that, A is closed, and B is compact (e.g. a point), then there exists a strict separating hyperplane $(x^*, \alpha) \in X^* \times \mathbb{R}$ such that, there exists $\varepsilon > 0$,

$$\langle x^*, a \rangle + \varepsilon \leq \alpha \leq \langle x^*, b \rangle - \varepsilon \qquad \forall a, b \in A \times B.$$

Remark: these theorems require the Zorn Lemma which is equivalent to the axiom of choice.

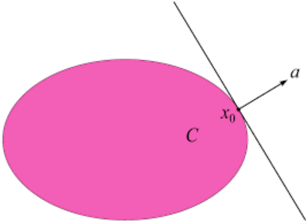
Supporting hyperplane

Theorem

Let $x_0 \notin \mathrm{ri}(C)$ and C convex. Then there exists $a \neq 0$ such that

$$a^\top x \geq a^\top x_0, \qquad \forall x \in C$$

If $x_0 \in C$, say that $H = \{x \mid a^\top x = a^\top x_0\}$ is a supporting hyperplane of C at x_0.



♣ Exercise: prove this theorem
Remark: there can be more than one supporting hyperplane at a given point.

Convex set as intersection of half-spaces

- The closed convex hull of $C \subset X$, denoted $\overline{\text{conv}}(C)$ is the smallest closed convex set containing C.
- $\overline{\text{conv}}(C)$ is the intersection of all the half-spaces containing C.
- A polyhedron is a finite intersection of half-spaces while a convex set is a possibly non-finite intersection of half-spaces.

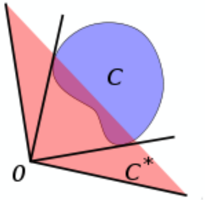
Dual and normal cones

- Let $C \subset \mathbb{R}^n$ be a set. We define its dual cone by

$$C^{\oplus} := \{x \mid x^\top c \geq 0, \quad \forall c \in C\}$$

- For any set C, $C^{\oplus}$ is a closed convex cone.
- The normal cone of C at x_0 is

$$N_C(x_0) := \{\lambda \in E \mid \lambda^\top (x - x_0) \leq 0, \quad \forall x \in C\}$$



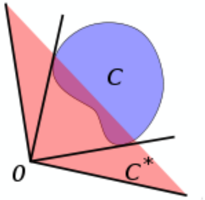
Dual and normal cones

- Let $C \subset \mathbb{R}^n$ be a set. We define its dual cone by

$$C^{\oplus} := \{x \mid x^\top c \geq 0, \quad \forall c \in C\}$$

- For any set C, $C^{\oplus}$ is a closed convex cone.
- The normal cone of C at x_0 is

$$N_C(x_0) := \{\lambda \in E \mid \lambda^\top (x - x_0) \leq 0, \quad \forall x \in C \}$$



Examples

- The positive orthant $K = \mathbb{R}^n_+$ is a self dual cone, that is $K^\oplus = K$.
- In the space of symetric matrices $S_n(\mathbb{R})$, with the scalar product $\langle A, B\rangle = \mathrm{tr}(AB)$, the set of positive semidefinite matrices $K = S_n^+(\mathbb{R})$ is self dual.
- Let $\|\cdot\|$ be a norm. The cone $K = \{(x,t) \mid \|x\| \leq t\}$ has for dual $K^\oplus = \{(\lambda, z) \mid \|\lambda\|_\star \leq z\}$, where $\|\lambda\|_\star := \sup_{x:\|x\|\leq 1} \lambda^\top x$.

♠ Exercise: prove these results

Some basic properties

Let $K \subset \mathbb{R}^n$ be a cone.

- $K^\oplus$ is closed convex.
- $K_1 \subset K_2$ implies $K_2^\oplus \subset K_1^\oplus$
- $K^{\oplus\oplus} = \overline{\mathrm{conv}}\, K$

♣ Exercise: Prove these results

Video ressources

https://www.youtube.com/watch?v=P3W_wFZ2kUo

Contents

Functions with non finite values ♡

- It is very useful in optimization to allow functions to take non-finite values, that is to take values in $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$.
- If both $-\infty$ and $+\infty$ are allowed be very careful of each addition !
- Let $f : X \to \bar{\mathbb{R}}$. We define
  - The epigraph of f as

$$\mathrm{epi}(f) := \{(x,t) \in X \times \mathbb{R} \quad | \quad f(x) \leq t\}$$

  - the domain of f as

$$\mathrm{dom}(f) := \{x \in X \quad | \quad f(x) < +\infty\}.$$

  - The sublevel set of level α

$$lev_\alpha(f) := \{x \in X \quad | \quad f(x) \leq \alpha\}.$$

- f is said to be lower semi continuous (l.s.c.) if $\mathrm{epi}(f)$ is closed.
- f is said to be proper if it never takes value $-\infty$, has a non-empty domain (at least one finite value).

Functions with non finite values ♡

- It is very useful in optimization to allow functions to take non-finite values, that is to take values in $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$.
- If both $-\infty$ and $+\infty$ are allowed be very careful of each addition !
- Let $f : X \to \bar{\mathbb{R}}$. We define
  - The epigraph of f as
$$\mathrm{epi}(f) := \{(x,t) \in X \times \mathbb{R} \quad | \quad f(x) \leq t \}$$
  - the domain of f as
$$\mathrm{dom}(f) := \{x \in X \quad | \quad f(x) < +\infty\}.$$
  - The sublevel set of level α
$$lev_\alpha(f) := \{x \in X \quad | \quad f(x) \leq \alpha\}.$$
- f is said to be lower semi continuous (l.s.c.) if $\mathrm{epi}(f)$ is closed.
- f is said to be proper if it never takes value $-\infty$, has a non-empty domain (at least one finite value).

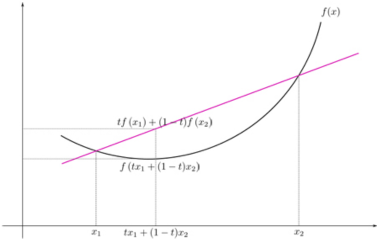
Functions with non finite values ♡

- It is very useful in optimization to allow functions to take non-finite values, that is to take values in $\bar{\mathbb{R}} := \mathbb{R} \cup \ \{-\infty, +\infty\}$.
- If both $-\infty$ and $+\infty$ are allowed be very careful of each addition !
- Let $f : X \to \bar{\mathbb{R}}$. We define
  - The epigraph of f as
$$\mathrm{epi}(f) := \{(x,t) \in X \times \mathbb{R} \quad | \quad f(x) \leq t \ \}$$
  - the domain of f as
$$\mathrm{dom}(f) := \{x \in X \quad | \quad f(x) < +\infty\}.$$
  - The sublevel set of level α
$$lev_\alpha(f) := \{x \in X \quad | \quad f(x) \leq \alpha\}.$$
- f is said to be lower semi continuous (l.s.c.) if $\mathrm{epi}(f)$ is closed.
- f is said to be proper if it never takes value $-\infty$, has a non-empty domain (at least one finite value).

Convex function ♡

- A function $f : X \to \bar{\mathbb{R}}$ is convex if its epigraph is convex.
- $f : X \to \mathbb{R} \cup \{+\infty\}$ is convex iff

$$\forall t \in [0,1], \forall x, y \in X,$$
$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$



- f is concave if $-f$ is convex.

Basic properties ♡

- If f, g convex, $t > 0$, then $tf + g$ is convex.
- If f convex non-decreasing, g convex, then $f \circ g$ convex.
- If f convex and a affine, then $f \circ a$ is convex.
- If $(f_i)_{i \in I}$ is a family of convex functions, then $\sup_{i \in I} f_i$ is convex.
- The domain and the sublevel sets of a convex function are convex.
- A convex function is always above its tangents.

♣ Exercise: Prove these results.

Basic properties

♡

- If f, g convex, $t > 0$, then $tf + g$ is convex.
- If f convex non-decreasing, g convex, then $f \circ g$ convex.
- If f convex and a affine, then $f \circ a$ is convex.
- If $(f_i)_{i \in I}$ is a family of convex functions, then $\sup_{i \in I} f_i$ is convex.
- The domain and the sublevel sets of a convex function are convex.
- A convex function is always above its tangents.

♣ Exercise: Prove these results.

Theorem (Jensen inequality)

Let f be a convex function and $\boldsymbol{X}$ an integrable random variable. Then we have

$$f(\mathbb{E}[\boldsymbol{X}]) \leq \mathbb{E}[f(\boldsymbol{X})].$$

Convex function: regularity ♡

Consider a convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$.

- f is continuous (on $\mathbb{R}^n$) if and only if $\mathrm{dom}(f) = \mathbb{R}^n$ (i.e., if it is finite everywhere)
- f is continuous on the interior of its domain
- f is lower-semicontinuous if and only if the domain is closed and the restriction of f to its domain is continuous

Convex functions: strict and strong convexity ♡

- $f : X \to \mathbb{R} \cup \{+\infty\}$ is strictly convex iff

$$\forall t \in]0,1[, \quad \forall x, y \in X, \qquad f(tx + (1-t)y) < tf(x) + (1-t)f(y)$$

- $f : X \to \mathbb{R} \cup \{+\infty\}$ is α-convex iff $\forall t \in]0,1[, \quad \forall x, y \in X,$

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \frac{1}{2}\alpha t(1-t)\|x-y\|^2$$

- If $f \in C^1(\mathbb{R}^n)$
  - $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0$ iff f convex
  - if strict inequality holds, then f strictly convex
  - $f : X \to \mathbb{R} \cup \{+\infty\}$ is α-convex iff $\forall x, y \in X$

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\alpha}{2}\|y - x\|^2$$

- If $f \in C^2(\mathbb{R}^n)$,
  - $\nabla^2 f \succeq 0$ iff f convex
  - if $\nabla^2 f \succ 0$ then f strictly convex
  - if $\nabla^2 f \succeq \alpha I$ then f is α-convex

Convex functions: strict and strong convexity ♡

- $f : X \to \mathbb{R} \cup \{+\infty\}$ is strictly convex iff

$$\forall t \in]0,1[, \quad \forall x, y \in X, \qquad f(tx + (1-t)y) < tf(x) + (1-t)f(y)$$

- $f : X \to \mathbb{R} \cup \{+\infty\}$ is α-convex iff $\forall t \in]0,1[, \quad \forall x, y \in X,$

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \frac{1}{2}\alpha t(1-t)\|x-y\|^2$$

- If $f \in C^1(\mathbb{R}^n)$
  - $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0$ iff f convex
  - if strict inequality holds, then f strictly convex
  - $f : X \to \mathbb{R} \cup \{+\infty\}$ is α-convex iff $\forall x, y \in X$

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\alpha}{2}\|y - x\|^2$$

- If $f \in C^2(\mathbb{R}^n)$,
  - $\nabla^2 f \succeq 0$ iff f convex
  - if $\nabla^2 f \succ 0$ then f strictly convex
  - if $\nabla^2 f \succeq \alpha I$ then f is α-convex

Convex functions: strict and strong convexity ♡

- $f : X \to \mathbb{R} \cup \{+\infty\}$ is strictly convex iff

$$\forall t \in]0,1[, \quad \forall x, y \in X, \qquad f(tx + (1-t)y) < tf(x) + (1-t)f(y)$$

- $f : X \to \mathbb{R} \cup \{+\infty\}$ is α-convex iff $\forall t \in]0,1[, \quad \forall x, y \in X,$

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \frac{1}{2}\alpha t(1-t)\|x-y\|^2$$

- If $f \in C^1(\mathbb{R}^n)$
  - $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0$ iff f convex
  - if strict inequality holds, then f strictly convex
  - $f : X \to \mathbb{R} \cup \{+\infty\}$ is α-convex iff $\forall x, y \in X$

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\alpha}{2}\|y - x\|^2$$

- If $f \in C^2(\mathbb{R}^n)$,
  - $\nabla^2 f \succcurlyeq 0$ iff f convex
  - if $\nabla^2 f \succ 0$ then f strictly convex
  - if $\nabla^2 f \succcurlyeq \alpha I$ then f is α-convex

Important examples

- The indicator function of a set $C \subset X$,

$$\mathbb{I}_C(x) := \begin{cases} 0 & \text{if } x \in C \\ +\infty & \textit{otherwise} \end{cases}$$

  is convex iff C is convex.
- $x \mapsto e^{ax}$ is convex for any $a \in \mathbb{R}$
- $x \mapsto \|x\|^q$ is convex for $q \geq 1$ and any norm
- $x \mapsto \ln(x)$ is concave
- $x \mapsto x\ln(x)$ is convex
- $x \mapsto \ln(\sum_{i=1}^n e^{x_i})$ is convex

Contents

Convex optimization problem ♡

$$\underset{x \in C}{\text{Min}} \quad f(x)$$

Where C is closed convex and f convex finite valued, is a convex optimization problem.

- If C is compact and f proper lsc, then there exists an optimal solution.
- If f is proper lsc and coercive, then there exists an optimal solution.
- The set of optimal solutions is convex.
- If f is strictly convex the minimum (if it exists) is unique.
- If f is α-convex the minimum exists and is unique.

♣ Exercise: Prove these results.

Optimality conditions ♡

Note that minimizing f over C or minimizing $f + \mathbb{I}_C$ over X is the same thing.

We consider the (unconstrained) optimization problem

$$\operatorname*{Min}_{x \in X} \quad f(x),$$

with $x^\sharp$ an optimal solution and f not necessarily convex.

- If f is differentiable, then $\nabla f(x^\sharp) = 0$.
- If f is twice differentiable, then $\nabla^2 f(x^\sharp) \succeq 0$.
- If f is twice differentiable and $\nabla^2 f(x_0) \succ 0$ then x_0 is a local minimum.

If, in addition, f is convex then $\nabla f(x) = 0$ is a sufficient optimality condition.

Optimality conditions ♡

Note that minimizing f over C or minimizing $f + \mathbb{I}_C$ over X is the same thing.
We consider the (unconstrained) optimization problem

$$\underset{x \in X}{\text{Min}} \qquad f(x),$$

with $x^\sharp$ an optimal solution and f not necessarily convex.

- If f is differentiable, then $\nabla f(x^\sharp) = 0$.
- If f is twice differentiable, then $\nabla^2 f(x^\sharp) \succeq 0$.
- If f is twice differentiable and $\nabla^2 f(x_0) \succ 0$ then x_0 is a local minimum.

If, in addition, f is convex then $\nabla f(x) = 0$ is a sufficient optimality condition.

Optimality conditions

♡

Note that minimizing f over C or minimizing $f + \mathbb{I}_C$ over X is the same thing.
We consider the (unconstrained) optimization problem

$$\underset{x \in X}{\text{Min}} \qquad f(x),$$

with $x^\sharp$ an optimal solution and f not necessarily convex.

- If f is differentiable, then $\nabla f(x^\sharp) = 0$.
- If f is twice differentiable, then $\nabla^2 f(x^\sharp) \succeq 0$.
- If f is twice differentiable and $\nabla^2 f(x_0) \succ 0$ then x_0 is a local minimum.

If, in addition, f is convex then $\nabla f(x) = 0$ is a sufficient optimality condition.

Optimality conditions ♡

Note that minimizing f over C or minimizing $f + \mathbb{I}_C$ over X is the same thing.
We consider the (unconstrained) optimization problem

$$\underset{x \in X}{\text{Min}} \qquad f(x),$$

with $x^\sharp$ an optimal solution and f not necessarily convex.

- If f is differentiable, then $\nabla f(x^\sharp) = 0$.
- If f is twice differentiable, then $\nabla^2 f(x^\sharp) \succeq 0$.
- If f is twice differentiable and $\nabla^2 f(x_0) \succ 0$ then x_0 is a local minimum.

If, in addition, f is convex then $\nabla f(x) = 0$ is a sufficient optimality condition.

Contents

Partial infimum

♡

Let f be a convex function and C a convex set. The function

$$g : x \mapsto \inf_{y \in C} f(x, y)$$

is convex.

♠ Exercise: Prove this result.

♣ Exercise: Prove that the function distance to a convex set C defined by

$$d_C(x) := \inf_{c \in C} \|c - x\|$$

is convex.

Partial infimum ♡

Let f be a convex function and C a convex set. The function

$$g : x \mapsto \inf_{y \in C} f(x, y)$$

is convex.

♠ Exercise: Prove this result.

♣ Exercise: Prove that the function distance to a convex set C defined by

$$d_C(x) := \inf_{c \in C} \|c - x\|$$

is convex.

Perspective function

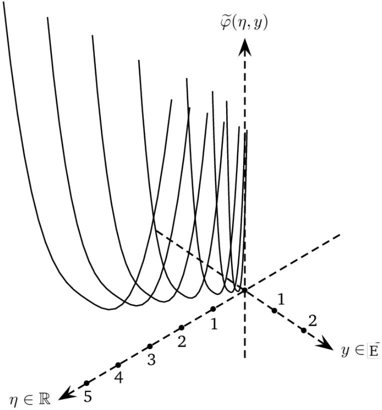
Let $\phi : E \to \bar{\mathbb{R}}$. The perspective of ϕ is defined as $\tilde{\phi} : \mathbb{R}_+^* \times E \to \mathbb{R}$ by

$$\tilde{\phi}(\eta, y) := \eta\phi(y/\eta).$$

Theorem

ϕ is convex iff $\tilde{\phi}$ is convex.

♠ Exercise: prove this result



Inf-Convolution

Let f and g be proper function from X to $\mathbb{R} \cup \{+\infty\}$. We define

$$f \square g : x \mapsto \inf_{y \in X} f(y) + g(x - y)$$

♣ Exercise: Show that

- $f \square g = g \square f$
- If f and g are convex then so is $f \square g$

Contents

Subdifferential of convex function

Let X be an Hilbert space, $f : X \to \bar{\mathbb{R}}$ convex.

- The subdifferential of f at $x \in \mathrm{dom}(f)$ is the set of slopes of all affine minorants of f exact at x:

$$\partial f(x) := \Big\{ \lambda \in X \quad | \quad f(\cdot) \geq \langle \lambda, \cdot - x \rangle + f(x) \Big\}.$$

- If f is derivable at x then

$$\partial f(x) = \big\{ \nabla f(x) \big\}.$$

Examples

◇

- If $f : x \mapsto |x|$, then

$$\partial f(x) = \begin{cases} -1 & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

- If C is convex then, for $x \in C$, $\partial(\mathbb{I}_C)(x) = N_C(x)$
  ♣ Exercise: Prove it.
- If f_1 and f_2 are convex and differentiable. Define $f = \max(f_1, f_2)$. Then
  - if $f_1(x) > f_2(x)$, $\partial f(x) = \{\nabla f_1(x)\}$
  - if $f_1(x) < f_2(x)$, $\partial f(x) = \{\nabla f_2(x)\}$;
  - if $f_1(x) = f_2(x)$, $\partial f(x) = \overline{\mathrm{conv}}(\{\nabla f_1(x), \nabla f_2(x)\})$.

Subdifferential calculus

Let f_1 and f_2 be proper convex functions.

Theorem

We have

$$\partial(f_1)(x) + \partial(f_2)(x) \subset \partial(f_1 + f_2)(x), \qquad \forall x$$

Further if $\mathrm{ri}(\mathrm{dom}(f_1)) \cap \mathrm{ri}(\mathrm{dom}(f_2)) \neq \emptyset$ *then*

$$\partial(f_1)(x) + \partial(f_2)(x) = \partial(f_1 + f_2)(x), \qquad \forall x$$

When f_i *is polyhedral you can replace* $\mathrm{ri}(\mathrm{dom}(f_i))$ *by* $\mathrm{dom}(f_i)$ *in the condition.*

Subdifferential calculus

Let f_1 and f_2 be proper convex functions.

Theorem

We have

$$\partial(f_1)(x) + \partial(f_2)(x) \subset \partial(f_1 + f_2)(x), \qquad \forall x$$

Further if $\mathrm{ri}(\mathrm{dom}(f_1)) \cap \mathrm{ri}(\mathrm{dom}(f_2)) \neq \emptyset$ *then*

$$\partial(f_1)(x) + \partial(f_2)(x) = \partial(f_1 + f_2)(x), \qquad \forall x$$

When f_i *is polyhedral you can replace* $\mathrm{ri}(\mathrm{dom}(f_i))$ *by* $\mathrm{dom}(f_i)$ *in the condition.*

Theorem

If f *is convex and* $a : x \mapsto Ax + b$ *with* $Im(a) \cap \mathrm{ri}(\mathrm{dom}(f)) \neq \emptyset$*, then*

$$\partial(f \circ a)(x) = A^\top \partial f(Ax + b).$$

First order optimality conditions

Theorem

Let $f : X \mapsto \mathbb{R} \cup \{+\infty\}$ *be a convex function (not necessarily) differentiable.* $x^\sharp$ *is a minimizer of* f *if and only if* $0 \in \partial f(x^\sharp)$.

First order optimality conditions

Theorem

Let $f : X \mapsto \mathbb{R} \cup \{+\infty\}$ *be a convex function (not necessarily) differentiable.* $x^\sharp$ *is a minimizer of* f *if and only if* $0 \in \partial f(x^\sharp)$.

Theorem

Let f *be a proper convex function and* C *a closed non-empty convex set such that* $\mathrm{ri}(C) \cap \mathrm{ri}(\mathrm{dom}(f)) \neq \emptyset$ *then* $x^\sharp$ *is an optimal solution to*

$$\min_{x \in C} \qquad f(x)$$

iff

$$0 \in \partial f(x^\sharp) + N_C(x^\sharp),$$

iff

$$\exists \lambda \in \partial f(x^\sharp), \quad \lambda \in -N_C(x^\sharp).$$

Normal cone, Tangent cone and optimality

Let C be a convex set. We define the tangent cone of $C \subset \mathbb{R}^n$ at point $x \in C$, as the set of directions in which you can move from x while staying in C for some time, that is

$$T_C(x) := \left\{ \lambda(y - x) \;\middle|\; y \in C, \quad \lambda \in \mathbb{R}^+ \right\}$$

In particular, $T_C(x) = \mathbb{R}^n$ iff $x \in int(C)$.

♣ Exercise: Prove that $[T_C(x)]^{\oplus} = -N_C(x)$.

Normal cone, Tangent cone and optimality

Let C be a convex set. We define the tangent cone of $C \subset \mathbb{R}^n$ at point $x \in C$, as the set of directions in which you can move from x while staying in C for some time, that is

$$T_C(x) := \left\{ \lambda(y - x) \;\middle|\; y \in C, \quad \lambda \in \mathbb{R}^+ \right\}$$

In particular, $T_C(x) = \mathbb{R}^n$ iff $x \in int(C)$.

♣ Exercise: Prove that $[T_C(x)]^{\oplus} = -N_C(x)$.

Partial infimum

Let $f : X \times Y \to \bar{\mathbb{R}}$ be a jointly convex and proper function, and define

$$v(x) = \inf_{y \in Y} f(x, y)$$

then v is convex.
If v is proper, and $v(x) = f(x, y^\sharp(x))$ then

$$\partial v(x) = \{ g \in X \quad | \quad (g, 0) \in \partial f(x, y^\sharp(x)) \}$$

proof:

$$\begin{aligned} g \in \partial v(x) \quad &\Leftrightarrow \quad \forall x', \qquad v(x') \geq v(x) + \langle g, x' - x \rangle \\ &\Leftrightarrow \quad \forall x', y' \quad f(x', y') \geq f(x, y^\sharp(x)) + \left\langle \begin{pmatrix} g \\ 0 \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} - \begin{pmatrix} x \\ y^\sharp(x) \end{pmatrix} \right\rangle \\ &\Leftrightarrow \quad \begin{pmatrix} g \\ 0 \end{pmatrix} \in \partial f(x, y^\sharp(x)) \end{aligned}$$

Partial infimum

Let $f : X \times Y \to \bar{\mathbb{R}}$ be a jointly convex and proper function, and define

$$v(x) = \inf_{y \in Y} f(x, y)$$

then v is convex.
If v is proper, and $v(x) = f(x, y^\sharp(x))$ then

$$\partial v(x) = \left\{ g \in X \quad | \quad (g, 0) \in \partial f(x, y^\sharp(x)) \right\}$$

proof:

$$\begin{aligned} g \in \partial v(x) \quad & \Leftrightarrow \quad \forall x', \qquad v(x') \geq v(x) + \langle g, x' - x \rangle \\ & \Leftrightarrow \quad \forall x', y' \quad f(x', y') \geq f(x, y^\sharp(x)) + \left\langle \begin{pmatrix} g \\ 0 \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} - \begin{pmatrix} x \\ y^\sharp(x) \end{pmatrix} \right\rangle \\ & \Leftrightarrow \quad \begin{pmatrix} g \\ 0 \end{pmatrix} \in \partial f(x, y^\sharp(x)) \end{aligned}$$

Partial infimum

Let $f : X \times Y \to \bar{\mathbb{R}}$ be a jointly convex and proper function, and define

$$v(x) = \inf_{y \in Y} f(x, y)$$

then v is convex.
If v is proper, and $v(x) = f(x, y^\sharp(x))$ then

$$\partial v(x) = \{ g \in X \quad | \quad (g, 0) \in \partial f(x, y^\sharp(x)) \}$$

proof:

$$\begin{aligned} g \in \partial v(x) \quad &\Leftrightarrow \quad \forall x', \qquad v(x') \geq v(x) + \langle g, x' - x \rangle \\ &\Leftrightarrow \quad \forall x', y' \quad f(x', y') \geq f(x, y^\sharp(x)) + \left\langle \begin{pmatrix} g \\ 0 \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} - \begin{pmatrix} x \\ y^\sharp(x) \end{pmatrix} \right\rangle \\ &\Leftrightarrow \quad \begin{pmatrix} g \\ 0 \end{pmatrix} \in \partial f(x, y^\sharp(x)) \end{aligned}$$

Convex function: regularity

- Assume f convex, then f is continuous on the relative interior of its domain, and Lipschitz on any compact contained in the relative interior of its domain.
- A proper convex function is subdifferentiable on the relative interior of its domain.
- If f is convex, it is L-Lipschitz iff $\partial f(x) \subset B(0, L), \quad \forall x \in \mathrm{dom}(f)$

Contents

Fenchel transform

Let X be a Hilbert space, $f : X \to \bar{\mathbb{R}}$ be a proper function.

- The Fenchel transform of f, is $f^\star : X \to \bar{\mathbb{R}}$ with

$$f^\star(\lambda) := \sup_{x \in X} \langle \lambda, x \rangle - f(x).$$

- $f^\star$ is convex lsc as the supremum of affine functions.
- $f \leq g$ implies that $f^\star \geq g^\star$.
- If f is proper convex lsc, then $f^{\star\star} = f$, otherwise $f^{\star\star} \leq f$.

♣ Exercise: Prove the first two points

Fenchel transform

Let X be a Hilbert space, $f : X \to \bar{\mathbb{R}}$ be a proper function.

- The Fenchel transform of f, is $f^\star : X \to \bar{\mathbb{R}}$ with

$$f^\star(\lambda) := \sup_{x \in X} \langle \lambda, x \rangle - f(x).$$

- $f^\star$ is convex lsc as the supremum of affine functions.
- $f \leq g$ implies that $f^\star \geq g^\star$.
- If f is proper convex lsc, then $f^{\star\star} = f$, otherwise $f^{\star\star} \leq f$.

♣ Exercise: Prove the first two points

Fenchel transform

Let X be a Hilbert space, $f : X \to \bar{\mathbb{R}}$ be a proper function.

- The Fenchel transform of f, is $f^\star : X \to \bar{\mathbb{R}}$ with

$$f^\star(\lambda) := \sup_{x \in X} \langle \lambda, x \rangle - f(x).$$

- $f^\star$ is convex lsc as the supremum of affine functions.
- $f \leq g$ implies that $f^\star \geq g^\star$.
- If f is proper convex lsc, then $f^{\star\star} = f$, otherwise $f^{\star\star} \leq f$.

♣ Exercise: Prove the first two points

Fenchel transform

Let X be a Hilbert space, $f : X \to \bar{\mathbb{R}}$ be a proper function.

- The Fenchel transform of f, is $f^\star : X \to \bar{\mathbb{R}}$ with

$$f^\star(\lambda) := \sup_{x \in X} \langle \lambda, x \rangle - f(x).$$

- $f^\star$ is convex lsc as the supremum of affine functions.
- $f \leq g$ implies that $f^\star \geq g^\star$.
- If f is proper convex lsc, then $f^{\star\star} = f$, otherwise $f^{\star\star} \leq f$.

♣ Exercise: Prove the first two points

Fenchel transform and subdifferential

- By definition $f^\star(\lambda) \geq \langle \lambda, x\rangle - f(x)$ for all x,
- thus we always have (Fenchel-Young) $f(x) + f^\star(\lambda) \geq \langle \lambda, x\rangle$.
- Recall that $\lambda \in \partial f(x)$ iff,

$$f(x') \geq f(x) + \langle \lambda, x' - x\rangle, \qquad \forall x'$$

iff

$$\langle \lambda, x\rangle - f(x) \geq \langle \lambda, x'\rangle - f(x') \qquad \forall x'$$

that is

$$\lambda \in \partial f(x) \Leftrightarrow x \in \arg\max_{x' \in X} \left\{ \langle \lambda, x'\rangle - f(x') \right\} \Leftrightarrow f(x) + f^\star(\lambda) = \langle \lambda, x\rangle$$

- If f proper convex lsc

$$\lambda \in \partial f(x) \quad \Longleftrightarrow \quad x \in \partial f^\star(\lambda).$$

Fenchel transform and subdifferential

- By definition $f^\star(\lambda) \geq \langle \lambda, x \rangle - f(x)$ for all x,
- thus we always have (Fenchel-Young) $f(x) + f^\star(\lambda) \geq \langle \lambda, x \rangle$.
- Recall that $\lambda \in \partial f(x)$ iff,

$$f(x') \geq f(x) + \langle \lambda, x' - x \rangle, \qquad \forall x'$$

iff

$$\langle \lambda, x \rangle - f(x) \geq \langle \lambda, x' \rangle - f(x') \qquad \forall x'$$

that is

$$\lambda \in \partial f(x) \Leftrightarrow x \in \arg\max_{x' \in X} \big\{ \langle \lambda, x' \rangle - f(x') \big\} \Leftrightarrow f(x) + f^\star(\lambda) = \langle \lambda, x \rangle$$

- If f proper convex lsc

$$\lambda \in \partial f(x) \quad \Longleftrightarrow \quad x \in \partial f^\star(\lambda).$$

Fenchel transform and subdifferential

- By definition $f^\star(\lambda) \geq \langle \lambda, x \rangle - f(x)$ for all x,
- thus we always have (Fenchel-Young) $f(x) + f^\star(\lambda) \geq \langle \lambda, x \rangle$.
- Recall that $\lambda \in \partial f(x)$ iff,

$$f(x') \geq f(x) + \langle \lambda, x' - x \rangle, \qquad \forall x'$$

iff

$$\langle \lambda, x \rangle - f(x) \geq \langle \lambda, x' \rangle - f(x') \qquad \forall x'$$

that is

$$\lambda \in \partial f(x) \Leftrightarrow x \in \arg\max_{x' \in X} \left\{ \langle \lambda, x' \rangle - f(x') \right\} \Leftrightarrow f(x) + f^\star(\lambda) = \langle \lambda, x \rangle$$

- If f proper convex lsc

$$\lambda \in \partial f(x) \quad \Longleftrightarrow \quad x \in \partial f^\star(\lambda).$$

Fenchel transform and subdifferential

- By definition $f^\star(\lambda) \geq \langle \lambda, x\rangle - f(x)$ for all x,
- thus we always have (Fenchel-Young) $f(x) + f^\star(\lambda) \geq \langle \lambda, x\rangle$.
- Recall that $\lambda \in \partial f(x)$ iff,

$$f(x') \geq f(x) + \langle \lambda, x' - x\rangle, \qquad \forall x'$$

iff

$$\langle \lambda, x\rangle - f(x) \geq \langle \lambda, x'\rangle - f(x') \qquad \forall x'$$

that is

$$\lambda \in \partial f(x) \Leftrightarrow x \in \arg\max_{x' \in X} \left\{\langle \lambda, x'\rangle - f(x')\right\} \Leftrightarrow f(x) + f^\star(\lambda) = \langle \lambda, x\rangle$$

- If f proper convex lsc

$$\lambda \in \partial f(x) \quad \Longleftrightarrow \quad x \in \partial f^\star(\lambda).$$

What you have to know

- What is a affine set, a convex set, a polyhedron, a (convex) cone
- What is a convex function, that it is above its tangents.
- Jensen inequality
- What is a convex optimization problem. That any local minimum is a global minimum.
- The necessary optimality condition $\nabla f(x^\sharp) \in [T_X(x^\sharp)]^\oplus$

What you really should know

- That you can separate convex sets with a linear function
- What is the positive dual of a cone
- Basic manipulations preserving convexity (sum, cartesian product, intersection, linear projection)
- What is the domain, the sublevel of a function f
- What is a lower semi-continuous function, a proper convex function
- Conditions of (strict, strong) convexity for differentiable functions
- The partial minimum of a convex function is convex
- The definition of the subdifferential.
- The definition of the Fenchel transform.
- The link between Fenchel transform and subdifferential.

What you have to be able to do

- Show that a set is convex
- Show that a function is (strictly, strongly) convex
- Go from constrained problem to unconstrained problem using the indicator function $\mathbb{I}_X$

What you should be able to do

- Compute dual cones
- Use advanced results (projection, partial infimum, perspective) to show that a function or a set is convex
- Compute the Fenchel transform of simple functions