Exercises: Duality

February 23, 2026

Exercise 1 (Dual formulation)**.** *Let $g : \mathbb{R}^n \to \mathbb{R}^m$. Show that*

1. $\mathbb{I}_{g(x)=0} = \sup_{\lambda \in \mathbb{R}^m} \lambda^\top g(x)$
2. $\mathbb{I}_{g(x)\leq 0} = \sup_{\lambda \in \mathbb{R}^m_+} \lambda^\top g(x)$
3. $\mathbb{I}_{g(x)\in C} = \sup_{\lambda \in -C^\oplus} \lambda^\top g(x)$ *where C is a closed convex cone, and* $C^\oplus := \{\lambda \in \mathbb{R}^m \mid \lambda^\top c \geq 0, \forall c \in C\}$.

Exercise 2 (Linear Programming)**.** *Consider the following linear problem (LP)*

$$(P) \quad \underset{x\geq 0}{\mathrm{Min}} \quad c^\top x \qquad s.t. \quad Ax = b$$

1. *Show that the dual of (P) is an LP.*
2. *Show that the dual of the dual of (P) is equivalent to (P).*

Exercise 3 (Quadratically Constrained Quadratic Programming)**.** *Consider the problem*

$$(QCQP) \quad \underset{x\in\mathbb{R}^n}{\mathrm{Min}} \quad \frac{1}{2}x^\top P_0 x + q_0^\top x + r_0$$
$$\frac{1}{2}x^\top P_i x + q_i^\top x + r_i \leq 0 \quad \forall i \in [m]$$

where $P_0 \in S^n_{++}$ and $P_i \in S^n_+$.

1. *Show by duality that, for $\mu \in \mathbb{R}^m_+$, there exists P_μ, q_μ and r_μ,such that $g(\mu) = -\frac{1}{2}q_\mu^\top P_\mu^{-1} q_\mu + r_\mu \leq val(P)$.*
2. *Give an easy condition under which $val(P) = \sup_{\mu\geq 0} g(\mu)$.*

Exercise 4 (Conic Programming)**.** *Let $K \subset \mathbb{R}^n$ be a closed convex pointed cone, and denote $x \preceq_K y$ iff $y \in x + K$. Consider the following program, with $A \in M_{m,n}$ and $b \in \mathbb{R}^m$.*

$$(P) \quad \underset{x\in\mathbb{R}^n}{\mathrm{Min}} \quad c^\top x \qquad s.t. \quad Ax = b, \quad x \preceq_K 0$$

1. *Show that (P) is a convex optimization problem.*
2. *Denote $\mathcal{L}(x, \lambda, \mu) = c^\top x + \lambda^\top(Ax - b) + \mu^\top x$. Show that* $\mathrm{val}(P) = \mathrm{Min}_{x\in\mathbb{R}^n} \sup_{\lambda\in\mathbb{R}^m, \mu\in K^\oplus} \mathcal{L}(x, \lambda, \mu)$.
3. *Give a dual problem to (P).*

Exercise 5 (Duality gap)**.** *Consider the following problem*

$$\underset{x\in\mathbb{R}, y\in\mathbb{R}^+_*}{\mathrm{Min}} \quad e^{-x} \qquad s.t. \quad x^2/y \leq 0$$

1. *Find the optimal solution of this problem.*
2. *Write and solve the (Lagrangian) dual problem. Is there a duality gap ?*

Exercise 6 (Two-way partitionning)**.** *Let $W \in S_n$ be a symmetric matrix, consider the following problem.*

$$(P) \quad \underset{x\in\mathbb{R}^n}{\mathrm{Min}} \quad x^\top W x \qquad s.t. \quad x_i^2 = 1 \qquad \forall i \in [n]$$

1. *Consider a set of n element that you want to partition in 2 subsets, with a cost $c_{i,j}$ if i and j are in the same set, and a cost $-c_{i,j}$ if they are in a different set. Justify that it can be solved by solving (P).*

2. *Is (P) a convex problem ?*

3. *Show that, for any $\lambda \in \mathbb{R}^n$ such that $W + \mathrm{diag}(\lambda) \succeq 0$, we have $\mathrm{val}(P) \geq -\sum \lambda_i$. Deduce a lower bound on $\mathrm{val}(P)$.*

Exercise 7 (Linear SVM : duality (hard-margin))**.** *Let $(x_i, y_i)_{i\in[n]}$ be labeled data with $x_i \in \mathbb{R}^d$ and $y_i \in \{-1, 1\}$. Consider the hard-margin SVM primal problem*

$$(P) \qquad \underset{w\in\mathbb{R}^d,\ b\in\mathbb{R}}{\mathrm{Min}} \quad \frac{1}{2}\|w\|_2^2 \qquad s.t. \quad y_i\left(w^\top x_i + b\right) \geq 1 \qquad \forall i \in [n].$$

1. *In which case can we guarantee strong duality for (P)?*

2. *Derive the Lagrangian dual and express an optimal primal solution $(w^\sharp, b^\sharp)$ in terms of an optimal dual solution.*

Exercise 8. *We consider the following problem.*

$$\begin{aligned} \underset{x_1,x_2}{\mathrm{Min}} \quad & x_1^2 + x_2^2 && (1)\\ s.t. \quad & (x_1-1)^2 + (x_2-1)^2 \leq 1 && (2)\\ & (x_1-1)^2 + (x_2+1)^2 \leq 1 && (3) \end{aligned}$$

1. *Classify this problem (After 5th course)*

2. *Find the optimal solution and value of this problem.*

3. *Write and solve the KKT equation for this problem.*

4. *Derive and solve the Lagrangian dual of this problem.*

5. *Do we have strong duality ? If yes, could we have known it from the start ? If not, can you comment on why ?*