# Exercises: Optimality conditions

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**Exercise 1.** *Solve the following optimization problem*

$$\begin{aligned} \underset{x,y\in\mathbb{R}^2}{\text{Min}} \quad & (x-1)^2+(y-2)^2 \\ & x \le y \\ & x+2y \le 2 \end{aligned}$$

**Exercise 2** (First order optimality condition)**.** *Consider, for $f$ differentiable,*

$$(P) \quad \underset{x\in\mathbb{R}^n}{\text{Min}} \quad f(x) \quad s.t. \quad x \in X$$

*Recall that*

$$T_X(x_0) = \left\{ d \in \mathbb{R}^n \mid \exists d_k \to d, \exists t_k \searrow 0, \; s.t.\; x_0 + t_k d_k \in X \right\}$$

*and* $K^\oplus = \left\{\lambda \mid \lambda^\top x \ge 0,\ \forall x \in K\right\}$.
*Show that*

1. *If $x_0$ is an optimal solution to $(P)$, then $\nabla f(x_0) \in [T_X(x_0)]^\oplus$.*
2. *If $f$ is convex, $X$ is closed convex, and $\nabla f(x_0) \in [T_X(x_0)]^\oplus$, then $x_0$ is an optimal solution to $(P)$.*

**Exercise 3.** *In the following cases, are the KKT conditions necessary / sufficient ?*

1.
$$\begin{aligned} \min_{x_1,x_2,x_3} \quad & 12x_1 - 5x_2 + 3x_3 \\ s.t. \quad & x_1 + 2x_2 - x_3 = 5 \\ & x_1 - x_2 \ge -2 \\ & 2x_1 - 4x_2 \le 12 \end{aligned}$$

2.
$$\begin{aligned} \min_{x_1,x_2,x_3} \quad & 4x_1^2 - x_1x_2 + x_2^2 - 12x_1 \\ s.t. \quad & x_1 - 2x_2 + x_3 = 5 \\ & x_1^2 + 3x_2^2 \le 10 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

3.
$$\begin{aligned} \min_{x_1,x_2,x_3} \quad & e^{x_1} - x_1x_2 + x_3^3 \\ s.t. \quad & \ln(e^{x_1-4x_2} + e^{x_1+x_3}) \le 2x_1 + 3 \\ & 2x_1^2 + x_2^2 \le 2 \end{aligned}$$

4.
$$\begin{aligned} \min_{x_1,x_2} \quad & -x_1 \\ s.t. \quad & -x_2 - (x_1-1)^3 \le 0 \\ & x_1, x_2 \ge 0 \end{aligned}$$

5.
$$\begin{aligned} \min_{x_1,x_2} \quad & -x_1 \\ s.t. \quad & x_2 - (x_1-1)^3 \le 0 \\ & x_1, x_2 \ge 0 \end{aligned}$$

**Exercise 4.** *Solve the following problem using first order optimality conditions*

$$\begin{aligned} \min_{x_1,x_2} \quad & -2(x_1-2)^2 - x_2^2 \\ s.t. \quad & x_1^2 + x_2^2 \le 25 \\ & x_1 \ge 0 \end{aligned}$$

**Exercise 5** (When KKT can fail without qualification)**.** *Consider the problem*

$$\min_{x\in\mathbb{R}} \; f(x) := x \qquad s.t. \qquad g(x) := x^2 \le 0.$$

1. *Find the (global) solution.*

2. *Write the KKT conditions at the solution and show that there is* no *Lagrange multiplier* $\mu \geq 0$ *satisfying them.*

3. *Compute the tangent cone* $T_X(x^\star)$ *and the linearized cone* $T_X^\ell(x^\star)$ *at the solution* $x^\star$ *(with* $X = \{x : \; g(x) \leq 0\}$*).*