Exercises: Convex analysis

February 23, 2026

Convex sets

Exercise 1 (Perspective function)**.** *Let* $P : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ *be the* perspective function *defined as* $P(x,t) = x/t$*, with* $\mathrm{dom}(P) = \mathbb{R}^n \times \mathbb{R}_+^*$*.*

1. *Show that the image by* P *of the segment* $[\binom{x}{s}, \binom{y}{t}]$ *is the segment* $[P(\binom{x}{s}), P(\binom{y}{t})]$*, i.e.* $P([\binom{x}{s}, \binom{y}{t}]) = [P(\binom{x}{s}), P(\binom{y}{t})]$*.*
2. *Show that, if* $C \subset \mathbb{R}^n \times \mathbb{R}_+^*$ *is convex, then* $P(C)$ *is convex.*
3. *Show that, if* $D \subset \mathbb{R}^n$*, then* $P^{-1}(D)$ *is convex.*

Exercise 2 (Dual cones)**.** *Recall that, for any set* $K \subset \mathbb{R}^n$*,* $K^\oplus := \{y \in \mathbb{R}^n \mid \forall x \in K,\ \langle y, x\rangle \geq 0\}$*. We say that* K *is self dual (which means that* K *is a closed convex cone) if* $K^\oplus = K$*.*

1. *Show that* $K = \mathbb{R}_+^n$ *is self dual.*
2. *We consider the set of symmetric matrices* S_n *with the scalar product* $\langle A, B\rangle = \mathrm{tr}(AB)$*. Show that* $K = S_n^+(\mathbb{R})$ *is self dual.*
3. *Let* $\|\cdot\|$ *be a norm, show that* $K = \{(x,t) \mid \|x\| \leq t\}$ *has for dual* $K^\oplus = \{(z,\lambda) \mid \|z\|_\star \leq \lambda\}$*, where* $\|z\|_\star := \sup_{x:\|x\|\leq 1} z^\top x$*.*

Exercise 3 (Normal cones of standard convex sets)**.** *Compute the normal cone* $N_C(x)$ *for the following closed convex sets:*

1. *(Euclidean ball)* $C = \{x \in \mathbb{R}^n :\ \|x\|_2 \leq 1\}$*.*
2. *(Simplex)* $\Delta = \{x \in \mathbb{R}^n :\ x \geq 0,\ \mathbf{1}^\top x = 1\}$*.*

Bonus. *For the ball, compute the tangent cone* $T_C(x)$ *and verify* $[T_C(x)]^\oplus = -N_C(x)$*.*

Convex functions

Exercise 4 (Recognizing convexity / strict / strong)**.** *For each function below, give: (i) its domain, (ii) whether it is convex, strictly convex, strongly convex on its domain (and provide a short justification).*
Let $A \in \mathbb{R}^{m\times n}$*,* $b \in \mathbb{R}^m$*,* $\lambda > 0$*, and assume* A *has full column rank when stated.*

1. $f_1(x) = \|Ax - b\|_2$*.*
2. $f_2(x) = \frac{1}{2}\|Ax - b\|_2^2$*.*
3. $f_3(x) = \frac{1}{2}\|x\|_2^2 + \lambda\|x\|_1$*.*
4. $f_4(x) = -\log(b - a^\top x)$ *with* $\mathrm{dom}(f_4) = \{x : a^\top x < b\}$*.*

Bonus. *For* f_2*, give a strong convexity modulus in terms of* A *(when* A *has full column rank).*

Exercise 5 (Moving average)**.** *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a convex function.*

1. *Show that,* $s \mapsto \int_0^1 f(st)dt$ *is convex.*
2. *Show that,* $\mathbb{R}_+^* \ni T \mapsto 1/T \int_0^T f(t)dt$ *is convex.*

Exercise 6 (Partial infimum)**.** *Let* $f : \mathbb{R}^n \times \mathbb{R}^m \to \bar{\mathbb{R}}$ *be a convex function and* $C \subset \mathbb{R}^m$ *a convex set. Show that the function*

$$g : x \mapsto \inf_{y\in C} f(x,y)$$

is convex.

Exercise 7 (log determinant)**.** *Let, for any* $X \in S_n$*,* $f(X) = \ln(\det(X))$ *for* $X \succ 0$*,* $-\infty$ *otherwise. Consider, for* $Z \succ 0$*, and* $V \in S_n$*, the function* $g : t \mapsto f(Z + tV)$*.*

1. *Show that* $g(t) = \sum_{i=1}^n \ln(1 + t\lambda_i) + f(Z)$*, where the* λ_i *are the eigenvalues of* $Z^{-1/2} V Z^{-1/2}$*.*

2. *Show that* g *is concave. Conclude that* f *is concave.*

Exercise 8 (Perspective function)**.** *Let* $\phi : E \to \mathbb{R} \cup \{+\infty\}$*. The perspective of* ϕ *is defined as* $\tilde{\phi} : \mathbb{R}_+^* \times E \to \mathbb{R}$ *by*

$$\tilde{\phi}(\eta, y) := \eta \phi(y/\eta).$$

Show that ϕ *is convex iff* $\tilde{\phi}$ *is convex.*

Fenchel transform and subdifferential

Exercise 9 (Norm)**.** *Let* $\|\cdot\|$ *be a norm on* $\mathbb{R}^n$ *and* $\|y\|_\star := \sup_{x:\|x\|\leq 1} y^\top x$ *be its dual norm. Let* $f : x \mapsto \|x\|$*. Compute* $f^\star$ *and* $\partial f(0)$*.*

Exercise 10 (Lasso)**.** *Let* $A \in \mathbb{R}^{m\times n}$*,* $b \in \mathbb{R}^m$*,* $\lambda > 0$ *and consider*

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax - b\|_2^2 + \lambda\|x\|_1.$$

1. *Show that the problem admits at least one solution, and is unique if* A *has full column rank.*

2. *Compute* $\partial\|x\|_1$ *and derive the optimality condition for a minimizer* $x^\sharp$*:*

$$0 \in A^\top(Ax^\sharp - b) + \lambda\,\partial\|x^\sharp\|_1.$$

3. *Prove the coordinate-wise characterization:*

$$x_i^\sharp \neq 0 \;\Rightarrow\; a_i^\top(Ax^\sharp - b) = -\lambda\,\mathrm{sign}(x_i^\sharp)$$

$$x_i^\sharp = 0 \;\Rightarrow\; |a_i^\top(Ax^\sharp - b)| \leq \lambda,$$

where a_i *is the* i*-th column of* A*.*

4. *If* $\lambda \geq \|A^\top b\|_\infty$*, then* $x^\sharp = 0$ *is optimal.*

5. *Interpretation: explain in one sentence why large* λ *promotes sparsity of* $x^\sharp$*.*

Exercise 11 (Fenchel calculus: indicator, support, and affine change)**.** *Let* $C \subset \mathbb{R}^n$ *be a nonempty closed convex set and define the indicator* $\mathbb{I}_C$ *and the support function*

$$\sigma_C(y) := \sup_{x\in C} y^\top x.$$

1. *Show that* $(\mathbb{I}_C)^\star = \sigma_C$*.*

2. *Compute* σ_C *for:*

   (a) $C = B_2(0,1) = \{x : \|x\|_2 \leq 1\}$*,*

   (b) $C = \{x : \|x\|_\infty \leq 1\}$*.*

3. *Let* $A \in \mathbb{R}^{m\times n}$*,* $b \in \mathbb{R}^m$*, and define* $f(x) = \mathbb{I}_C(Ax + b)$*. Give an expression for* $f^\star$ *(you may state the formula with the condition under which it holds).*

Exercise 12 (Log sum exp)**.** *We consider* $f(x) := \ln(\sum_{i=1}^n e^{x_i})$*.*

1. *Show that* f *is convex. Hint : recall Holder's inequality* $x^\top y \leq \|x\|_p\|y\|_q$ *for* $1/p + 1/q = 1$*.*

2. *Show that* $f^\star(y) = \sum_{i=1}^n y_i \ln(y_i)$ *if* $y \geq 0$ *and* $\sum_i y_i = 1$*,* $+\infty$ *otherwise.*