Exercises: optimization problem classes

Exercise 1 (Hyperbolic constraints as SOCP)**.**

1. *Show that, for all $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $z \in \mathbb{R}$,*

$$x^\top x \le yz, \quad y \ge 0, \quad z \ge 0$$

iff

$$\left\| \begin{pmatrix} 2x \\ y - z \end{pmatrix} \right\| \le y + z \quad y \ge 0, \quad z \ge 0$$

2. *Represent the following problem as an SOCP*

$$\begin{aligned} (P) \text{ Max} \quad & \left(\sum_{i=1}^{n} 1/(a_i^\top x - b) \right)^{-1} \\ s.t. \quad & Ax > b \end{aligned}$$

Answers:

1. Assume $y \ge 0, z \ge 0$, then $y + z \ge 0 \iff xy \ge 0$. We now assume $y + z \ge 0$, then we have

$$\begin{aligned} & \left\| \begin{pmatrix} 2x \\ y - z \end{pmatrix} \right\| \le y + z \\ \iff & \left\| \begin{pmatrix} 2x \\ y - z \end{pmatrix} \right\|^2 \le (y+z)^2 \\ \iff & 4x^\top x + (y-z)^2 \le y^2 + 2yz + z^2 \\ \iff & 4x^\top x \le 4yz \\ \iff & x^\top x \le yz \end{aligned}$$

2. Since $t \mapsto 1/t$ is decreasing (P) is equivalent to

$$\begin{aligned} \text{Min} \quad & \sum_{i=1}^{n} 1/(a_i^\top x - b) \\ s.t. \quad & Ax > b \end{aligned}$$

By adding the lift variables z_i, (P) is equivalent to the problem

$$\begin{aligned} \text{Min} \quad & \sum_{i=1}^{n} z_i \\ s.t. \quad & Ax > b \\ & z_i \ge 1/(a_i^\top x - b) \ge 0, \forall i \in [n] \end{aligned}$$

which is equivalent to

$$\begin{aligned} \text{Min} \quad & \sum_{i=1}^{n} z_i \\ s.t. \quad & Ax \ge b \\ & 1 \le z(a_i^\top x - b), \forall i \in [n] \\ & z_i \ge 0, \forall i \in [n] \end{aligned}$$

By question 1. it is equivalent to the following SOCP:

$$\begin{aligned} \text{Min} \quad & \sum_{i=1}^{n} z_i \\ s.t. \quad & -Ax \le -b \\ & \left\| \begin{pmatrix} 2 \\ a_i^\top x - b - z \end{pmatrix} \right\| \le a_i^\top x - b + z_i, \forall i \in [n] \\ & z_i \ge 0, \forall i \in [n] \end{aligned}$$

Exercise 2. *We consider a physical function Φ that is approximated as the superposition of multiple simple phenomenon (e.g. waves). Each simple phenomenon $p \in [P]$ is represented by a function $\Phi_p : \mathbb{R}^d \to \mathbb{R}$.*
We have data points $(x^k, y^k)_{k \in [n]}$, and want to find the Φ that match at best *the data while being a linear combination of Φ_p.*
Propose a least-square regression that answer this question.

Answers: We define the matrix $M \in \mathbb{R}^{n\times P}$ with coefficients $M_{k,p} = \Phi_p(x_k)$. We propose the following last square regression problem:

$$\underset{\alpha\in\mathbb{R}^P}{\text{Min}} \qquad \|M\alpha - y\|^2$$

Exercise 3. *Consider a chocolate manufacturing company that produces only two types of chocolate – A and B. Both the chocolates require Milk and Choco only. To manufacture each unit of A and B, the following quantities are required:*

- *Each unit of A requires 1 unit of Milk and 3 units of Choco*
- *Each unit of B requires 1 unit of Milk and 2 units of Choco*

The company kitchen has a total of 5 units of Milk and 12 units of Choco. On each sale, the company makes a profit of

- *6 per unit A sold*
- *5 per unit B sold.*

Model this as an LP.

Exercise 4. *A classical extension of the least-square problem, which has strong theoretical and practical intereset is the LASSO problem*

$$\underset{x\in\mathbb{R}^p}{\text{Min}} \qquad \|Ax - b\|^2 + \lambda\|x\|_1$$

Show that this problem can be cast as a QP problem.

Answers: The LASSO problem is equivalent to the following QP problem

$$\begin{aligned}
\underset{x\in\mathbb{R}^n, z\in\mathbb{R}^n}{\text{Min}} \qquad & x^\top A^\top Ax - 2b^\top Ax + \lambda\sum_{i=1}^n z_i \\
\text{s.t.} \qquad & x_i \leq z_i \\
& -x_i \leq z_i
\end{aligned}$$

Exercise 5. *Consider the following optimization problem.*

$$\begin{aligned}
\underset{x\in\mathbb{R}^n}{\text{Min}} \qquad & c^\top x \\
s.t. \qquad & Ax = b \\
& x_i \in \{0,1\} \qquad\qquad \forall i \in I
\end{aligned}$$

Write this problem as a QCQP. Is it convex ?

Answers: The constraint $x_i \in \{0,1\}$ is equivalent to $x_i(1-x_i) = 0$. We define $q_i := (0,\cdots,0,1,0,\cdots,0)$ as the vector with all coordinates equal to 0 except the ith which equals 1. We set $Q_i = 2diag(q_i) = 2q_iq_i^\top$. Then, this problem is equivalent to the following QCQP problem

$$\begin{aligned}
\underset{x\in\mathbb{R}^n}{\text{Min}} \qquad & c^\top x \\
\text{s.t.} \qquad & Ax = b \\
& \frac{1}{2}x^\top(2e_ie_i^\top)x - e_i^\top x \leq 0, \\
& -\frac{1}{2}x^\top(2e_ie_i^\top)x + e_i^\top x \leq 0.
\end{aligned}$$

It is not convex in the general case (i.e. if the admissible set is neither empty or reduced to a singleton) since the set $\{0,1\}^n$ is not convex. Remark that $-Q_i$ is not positive.

Exercise 6. *Consider a facility that plan to deliver product to clients by drone (thus in direct line). Assume that you have N clients, each with position (in $\mathbb{R}^2$) x_n. The drone make each time a direct travel from the facility location to the client. Assume that the drone have a maximum range of R, and that you want to minimize the average travel distance while being able to serve all of your clients.*
Model the problem of choosing the facility location as an SOCP.

Answers: We want to minimize the average travel distance $\frac{1}{N}\sum_{n=1}^N \|x_n - y\|$ from a center y to the clients (x_n) while being able to serve all of your clients. We modelize this by the problem

$$\begin{aligned}
\underset{y\in\mathbb{R}^2}{\text{Min}} \quad & \frac{1}{N}\sum_{n=1}^N \|x_n - y\| && (1) \\
\text{s.t.} \quad & \|x_n - y\| \leq R,\ \forall n \in [N] && (2)
\end{aligned}$$

By adding lift variables z_n, this equivalent to the following SOCP:

$$\begin{aligned}
\underset{y\in\mathbb{R}^2, z\in\mathbb{R}^N}{\text{Min}} \quad & \frac{1}{n}\sum_{n=1}^N z_n && (3) \\
\text{s.t.} \quad & \|x_n - y\| \leq R,\ \forall n \in [N] && (4) \\
& \|x_n - y\| \leq z_n,\ \forall n \in [N] && (5)
\end{aligned}$$

Exercise 7. *Consider the following robust linear program*

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{Min}} \quad & c^\top x \\ s.t. \quad & (a_i + R_i\delta_i)^\top x \leq b_i \quad \forall \|\delta_i\|_2 \leq 1, \quad \forall i \in [m] \end{aligned}$$

where R_i are positive real numbers. Write this problem as an SOCP.

Answers: The constraint $(a_i + R_i\delta_i)^\top x \leq b_i, \quad \forall \|\delta_i\|_2 \leq 1$ is equivalent to

$$\max_{\delta_i \mid \|\delta_i\|_2 \leq 1} (a_i + R_i\delta_i)^\top x \leq b_i$$

which is equivalent to

$$R_i \max_{\delta_i \mid \|\delta_i\|_2 \leq 1} \delta_i^\top x \leq b_i - a_i^\top x.$$

However, $\max_{\delta_i \mid \|\delta_i\|_2 \leq 1} \delta_i^\top x = \|x\|$. Indeed, this result is trivial for $x = 0$, for $x \neq 0$, we have that $\|x\|$ is an upper bound by Cauchy-Swhartz inequality which is attained for $\delta_i = \frac{x}{\|x\|}$. Thus, our problem is equivalent to the following SOCP:

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{Min}} \quad & c^\top x \\ \text{s.t.} \quad & R_i\|x\| \leq -a_i^\top x + b_i, \qquad \forall i \in [m] \end{aligned}$$

Exercise 8. *Let $F(\theta)$ be a symmetric matrix parametrized by $\theta \in \mathbb{R}^d$ whose coefficients are linear in θ. Model the problem of finding the parameter $\theta \in \Theta$, where Θ is a polyhedron, minimizing $\kappa(\theta)$ as an SDP.*
What happen if the coefficient of $F(\theta)$ are affine in θ ? Suggest a solution method ? (hard)

Exercise 9. *Consider a finite set $X = \{x_i\}_{i\in[n]}$, and $\mathcal{P}^+$ the set of probabilities on X. For $\mathbb{P}, \mathbb{Q} \in \mathcal{P}$, with* supp$(\mathbb{Q}) = X$, *we define the* Kullback-Leibler *divergence as*

$$d_{KL}(\mathbb{P}|\mathbb{Q}) = \sum_{i=1}^n p_i \ln(p_i/q_i)$$

where $p_i = \mathbb{P}(X = x_i)$ and $q_i = \mathbb{Q}(X = x_i)$.
Let X be 100 *equidistant points spanning in $[-1, 1]$. Let $\mathbb{Q}$ be uniform on X.*
We are looking for the probability $\mathbb{P}$ on X such that

- $\mathbb{E}_{\mathbb{P}}[\boldsymbol{X}] \in [-0.1, 0.1]$
- $\mathbb{E}_{\mathbb{P}}[\boldsymbol{X}^2] \in [0.5, 0.6]$
- $\mathbb{E}_{\mathbb{P}}[3\boldsymbol{X}^2 - 2\boldsymbol{X}] \in [-0.3, -0.2]$
- $\mathbb{P}(\boldsymbol{X} < 0) \in [0.3, 0.4]$

that minimize the Kullback-Leibler divergence from $\mathbb{Q}$.
Model this problem as an optimization problem. In which class does it belongs ?

Exercise 10. *Consider that you sell a given product over T days. The demand for each day is d_t. Having a quantity x_t of items in stock have a cost (per day) of cx_t. You can order, each day, a quantity q_t, and have to satisfy the demand.*
For each of the following variation : model the problem, explicit the class to which it belongs, and give the optimal solution if easily found.

1. *Without any further constraint / specifications.*
2. *There is an "ordering cost": each time you order, you have to pay a fix cost κ.*
3. *Instead of an "ordering cost" there is a maximum number of days at which you can order a replenishment.*