Exercises: Duality

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Exercise 1 (Dual formulation)**.** *Let $g : \mathbb{R}^n \to \mathbb{R}^m$. Show that*

1. $\mathbb{I}_{g(x)=0} = \sup_{\lambda \in \mathbb{R}^m} \lambda^\top g(x)$
2. $\mathbb{I}_{g(x)\leq 0} = \sup_{\lambda \in \mathbb{R}^m_+} \lambda^\top g(x)$
3. $\mathbb{I}_{g(x)\in C} = \sup_{\lambda \in -C^\oplus} \lambda^\top g(x)$ *where C is a closed convex cone, and $C^\oplus := \{\lambda \in \mathbb{R}^m \mid \lambda^\top c \geq 0, \forall c \in C\}$.*

Answers:

1. If $g(x) \neq 0$ there is $i \in [m]$ such that $g_i(x) \neq 0$, and we choose λ_i accordingly.
2. Same reasoning.
3. If $g(x) \in C$, $\lambda^\top g(x) \leq 0$, and $0 \in -C^\oplus$. If $g(x) \notin C$, by separation of the convex compact $\{g(x)\}$ from the closed convex set C there exists $\lambda \in \mathbb{R}^m$ such that $\lambda^\top g(x) > b > \lambda^\top c$ for all $c \in C$. As C is a cone, $tc \in C$ for all $t > 0$, and thus $\lambda \in -C^\oplus$. Further $b \geq 0$, thus $t\lambda^\top g(x) \to +\infty$ when $t \to \infty$.

Exercise 2 (Linear Programming)**.** *Consider the following linear problem (LP)*

$$(P) \quad \underset{x\geq 0}{\text{Min}} \quad c^\top x \qquad s.t. \quad Ax = b$$

1. *Show that the dual of (P) is an LP.*
2. *Show that the dual of the dual of (P) is equivalent to (P).*

Answers:

1. The dual of (P) is
$$(D) \quad \underset{\lambda}{\text{Max}} \quad -b^\top \lambda \qquad \text{s.t.} \quad A^\top \lambda + c \leq 0$$
2. Direct by computing the dual of (D).

Exercise 3 (Quadratically Constrained Quadratic Programming)**.** *Consider the problem*

$$(QCQP) \quad \underset{x\in\mathbb{R}^n}{\text{Min}} \quad \frac{1}{2}x^\top P_0 x + q_0^\top x + r_0$$
$$\frac{1}{2}x^\top P_i x + q_i^\top x + r_i \leq 0 \quad \forall i \in [m]$$

where $P_0 \in S^n_{++}$ and $P_i \in S^n_+$.

1. *Show by duality that, for $\mu \in \mathbb{R}^m_+$, there exists P_μ, q_μ and r_μ, such that $g(\mu) = -\frac{1}{2}q_\mu^\top P_\mu^{-1} q_\mu + r_\mu \leq val(P)$.*
2. *Give an easy condition under which $val(P) = \sup_{\mu\geq 0} g(\mu)$.*

Answers:

1. Simply write the dual function we get
$$P_\mu = P_0 + \sum_i \mu_i P_i, \quad q_\mu = q_0 + \sum_i \mu_i q_i$$
$$r_\mu = r_0 + \sum_i \mu_i r_i$$
2. The problem is convex, Slater's condition ensure constraint qualification, thus a condition would be the existence of a strictly satisfying all constraints.

Exercise 4 (Conic Programming)**.** *Let $K \subset \mathbb{R}^n$ be a closed convex pointed cone, and denote $x \preceq_K y$ iff $y \in x + K$. Consider the following program, with $A \in M_{m,n}$ and $b \in \mathbb{R}^m$.*

$$(P) \quad \underset{x\in\mathbb{R}^n}{\text{Min}} \quad c^\top x \qquad s.t. \quad Ax = b, \quad x \preceq_K 0$$

1. *Show that (P) is a convex optimization problem.*
2. *Denote $\mathcal{L}(x,\lambda,\mu) = c^\top x + \lambda^\top(Ax - b) + \mu^\top x$. Show that $\mathrm{val}(P) = \mathrm{Min}_{x\in\mathbb{R}^n} \sup_{\lambda\in\mathbb{R}^m,\mu\in K^\oplus} \mathcal{L}(x,\lambda,\mu)$.*
3. *Give a dual problem to (P).*

Answers:

1. $x \preceq_K 0$ iff $x \in -K$, and $-K$ is convex.
2. If $x \in -K$, for any $\mu \in K^\oplus$, $\mu^\top x \leq 0$, thus $\sup_{\mu\in K^\oplus} \mu^\top x = 0$. If $-x \notin K = K^{\oplus\oplus}$, there exists $\lambda \in K^\oplus$, such that $-x^\top\lambda < 0$, hence $\sup_{\mu\in K^\oplus} \mu^\top x = +\infty$. (or see ex 1).
3. By min-max duality we consider

$$\underset{\lambda\in\mathbb{R}^m,\mu\in K^\oplus}{\mathrm{Max}} -b^\top\lambda + \inf_{x\in\mathbb{R}^n} (A^\top\lambda + c + \mu)^\top x +$$

wich yields

$$(D) \quad \mathrm{Max} \quad -b^\top\lambda \qquad A^\top\lambda + c + \mu = 0 \quad \mu \in K^\oplus$$

Exercise 5 (Duality gap). *Consider the following problem*

$$\begin{aligned} \underset{x\in\mathbb{R},y\in\mathbb{R}_*^+}{\mathrm{Min}} \quad & e^{-x} \\ s.t. \quad & x^2/y \leq 0 \end{aligned}$$

1. *Find the optimal solution of this problem.*
2. *Write and solve the (Lagrangian) dual problem. Is there a duality gap ?*

Answers:

$$\mathcal{L}(x,y;\mu) = e^{-x} + \mu x^2/y$$

$$g(\mu) = \inf_{x\in\mathbb{R},y>0} e^{-x} + \mu x^2/y = 0$$

as the term inside the inf is positive, and choosing $x = t$, $y = t^3$ goes to 0 for all μ.

Exercise 6 (Two-way partitionning). *Let $W \in S_n$ be a symmetric matrix, consider the following problem.*

$$\begin{aligned} (P) \quad \underset{x\in\mathbb{R}^n}{\mathrm{Min}} \quad & x^\top W x \\ s.t. \quad & x_i^2 = 1 \qquad \forall i \in [n] \end{aligned}$$

1. *Consider a set of n element that you want to partition in 2 subsets, with a cost $c_{i,j}$ if i and j are in the same set, and a cost $-c_{i,j}$ if they are in a different set. Justify that it can be solved by solving (P).*
2. *Is (P) a convex problem ?*
3. *Show that, for any $\lambda \in \mathbb{R}^n$ such that $W + \mathrm{diag}(\lambda) \succeq 0$, we have $\mathrm{val}(P) \geq -\sum \lambda_i$. Deduce a lower bound on $\mathrm{val}(P)$.*

Answers:

1. The constraint ensures that $x_i \in \{-1,1\}$, each value representing one subset. We set $W_{i,j} = c_{i,j}$.
2. No, because the set of admissible points is $\{-1,1\}^n$.
3. The Lagrangian of (P) is

$$\begin{aligned} \mathcal{L}(x,\lambda) &= x^\top W x + \sum_{i=1}^n \lambda_i(x_i^2 - 1) \\ &= x^\top(W + diag(\lambda))x - \sum_{i=1}^n \lambda_i \end{aligned}$$

And we have,

$$\begin{aligned} g(\lambda) &= \inf_x x^\top(W + diag(\lambda))x - \sum_{i=1}^n \lambda_i \\ &= -\sum_{i=1}^n \lambda_i - \mathbb{I}_{W+diag(\lambda)\succeq 0} \leq val(P) \end{aligned}$$

Thus, if λ_{min} is the small eigenvalue of W we have $W + diag(\lambda) \geq 0$, and $val(P) \geq n\lambda_{min}$.

Exercise 7 (Linear SVM : duality (hard-margin)). *Let $(x_i, y_i)_{i\in[n]}$ be labeled data with $x_i \in \mathbb{R}^d$ and $y_i \in \{-1,1\}$. Consider the hard-margin SVM primal problem*

$$\begin{aligned} (P) \qquad \underset{w\in\mathbb{R}^d,\ b\in\mathbb{R}}{\mathrm{Min}} \quad & \frac{1}{2}\|w\|_2^2 \\ s.t. \quad & y_i\,(w^\top x_i + b) \geq 1 \qquad \forall i \in [n]. \end{aligned}$$

1. *In which case can we guarantee strong duality for (P)?*

2. *Derive the Lagrangian dual and express an optimal primal solution $(w^\sharp, b^\sharp)$ in terms of an optimal dual solution.*

Answers:

1. The problem is convex. Strong duality holds under Slater's condition, i.e., if there exists $(\bar{w}, \bar{b})$ such that $y_i(\bar{w}^\top x_i + \bar{b}) > 1$ for all $i \in [n]$ (strict feasibility, which holds for linearly separable data with a positive margin).

2. Introduce multipliers $\alpha \in \mathbb{R}^n_+$ for the constraints $1 - y_i(w^\top x_i + b) \leq 0$. The Lagrangian is

$$\mathcal{L}(w, b, \alpha) = \frac{1}{2}\|w\|_2^2 + \sum_{i=1}^n \alpha_i\big(1 - y_i(w^\top x_i + b)\big).$$

Stationarity in (w, b) gives

$$\nabla_w \mathcal{L} = w - \sum_{i=1}^n \alpha_i y_i x_i = 0 \quad \Rightarrow \quad w = \sum_{i=1}^n \alpha_i y_i x_i,$$

$$\partial_b \mathcal{L} = -\sum_{i=1}^n \alpha_i y_i = 0 \quad \Rightarrow \quad \sum_{i=1}^n \alpha_i y_i = 0.$$

Plugging w back into the Lagrangian yields the dual function

$$g(\alpha) = \sum_{i=1}^n \alpha_i - \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \alpha_i\alpha_j y_i y_j\, x_i^\top x_j,$$

and the dual problem is

$$(D) \qquad \underset{\alpha \in \mathbb{R}^n}{\text{Max}} \quad \sum_{i=1}^n \alpha_i - \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \alpha_i\alpha_j y_i y_j\, x_i^\top x_j$$

$$\text{s.t.} \quad \alpha_i \geq 0 \;\forall i \in [n], \qquad \sum_{i=1}^n \alpha_i y_i = 0.$$

Let $\alpha^\sharp$ be optimal. Then an optimal primal weight is

$$w^\sharp = \sum_{i=1}^n \alpha_i^\sharp y_i x_i.$$

Complementary slackness gives

$$\alpha_i^\sharp\big(1 - y_i((w^\sharp)^\top x_i + b^\sharp)\big) = 0 \quad \forall i.$$

For any index i with $\alpha_i^\sharp > 0$ (a support vector), we must have $y_i((w^\sharp)^\top x_i + b^\sharp) = 1$, hence

$$b^\sharp = y_i - (w^\sharp)^\top x_i \qquad \text{for any } i \text{ such that } \alpha_i^\sharp > 0.$$

Exercise 8. *We consider the following problem.*

$$\underset{x_1,x_2}{\text{Min}} \qquad x_1^2 + x_2^2 \tag{1}$$

$$s.t. \qquad (x_1-1)^2 + (x_2-1)^2 \leq 1 \tag{2}$$

$$(x_1-1)^2 + (x_2+1)^2 \leq 1 \tag{3}$$

1. *Classify this problem (After 5th course)*
2. *Find the optimal solution and value of this problem.*
3. *Write and solve the KKT equation for this problem.*
4. *Derive and solve the Lagrangian dual of this problem.*
5. *Do we have strong duality ? If yes, could we have known it from the start ? If not, can you comment on why ?*

Answers:

1. This is a convex QCQP
2. The only admissible point, and hence the optimal solution is $(1, 0)$, with value 1.
3. The Lagrangian is

$$\mathcal{L}(x,\lambda) = x_1^2 + x_2^2 + \lambda_1((x_1-1)^2 + (x_2-1)^2 - 1) + \lambda_2((x_1-1)^2 + (x_2+1)^2 - 1)$$

KKT condition are

- Gradient of Lagrangian is null :

$$2x_1 + 2\lambda_1(x_1-1) + 2\lambda_2(x_1-1) = 0$$
$$2x_2 + 2\lambda_1(x_2-1) + 2\lambda_2(x_2+1) = 0$$

- x is primal feasible : $(x_1-1)^2 + (x_2-1)^2 \leq 1$ and $(x_1-1)^2 + (x_2+1)^2 \leq 1$
- λ is dual feasible $\lambda_1 \geq 0$, $\lambda_2 \geq 0$.
- Complementary slackness:

$$\lambda_1 = 0 \quad \text{or} \quad (x_1-1)^2 + (x_2-1)^2 = 1$$
$$\lambda_2 = 0 \quad \text{or} \quad (x_1-1)^2 + (x_2+1)^2 = 1$$

x feasible is $x = (1, 0)$, which imply $2 = 0$ which is impossible. Thus there is no pair (x, λ) satisfying the KKT equations. The KKT equations fails to give the optimal solution because the constraints are not qualified.

4. The Lagrange dual function is

$$
\begin{aligned}
g(\lambda_1, \lambda_2) = & \inf_{x_1, x_2} \mathcal{L}(x, \lambda) \\
= & \inf_{x_1, x_2} (1 + \lambda_1 + \lambda_2)(x_1^2 + x_2)^2 \\
& - 2(\lambda_1 + \lambda_2)x_1 - 2(\lambda_1 - \lambda_2)x_2 + \lambda_1 + \lambda_2 \\
= & \lambda_1 + \lambda_2 - \frac{(\lambda_1 + \lambda_2)^2 + (\lambda_1 - \lambda_2)^2}{1 + \lambda_1 + \lambda_2} \\
& \text{if } 1 + \lambda_1 + \lambda_2 > 0
\end{aligned}
$$

The dual problem reads

$$
\begin{aligned}
\operatorname*{Max}_{\lambda} \quad & \frac{\lambda_1 + \lambda_2 - (\lambda_1 - \lambda_2)^2}{1 + \lambda_1 + \lambda_2} \\
\text{s.t.} \quad & \lambda_1 \geq 0, \lambda_2 \geq 0
\end{aligned}
$$

By symmetry the optimum is attained at $\lambda_1 = \lambda_2$, thus the dual reads

$$
\operatorname*{Max}_{\lambda_1 \geq 0} \quad \frac{2\lambda_1}{2\lambda_1 + 1}
$$

Which has value 1 and no solution.

5. The dual problem have the same value as the primal problem, thus we have strong duality.

   However there does not exist a dual multiplier, which is why there is no solution to the KKT equations.

   We could not guarantee the existence of a primal-dual optimal solution through KKT as the constraints were not qualified.