Exercises: Optimality conditions

February 23, 2026

Exercise 1. *Solve the following optimization problem*

$$\begin{aligned} \underset{x,y\in\mathbb{R}^2}{\text{Min}} \quad & (x-1)^2 + (y-2)^2 \\ & x \leq y \\ & x + 2y \leq 2 \end{aligned}$$

Answers: The problem is convex and qualified through Slater's condition (e.g. $(-1,0)$). Lagrangian

$$\begin{aligned} \mathcal{L}(x,y,\mu) =& (x-1)^2 + (y-2)^2 \\ & + \mu_1(x-y) + \mu_2(x+2y-2) \end{aligned}$$

KKT conditions

$$\begin{cases} 2(x-1) + \mu_1 + \mu_2 = 0 \\ 2(y-2) - \mu_1 + 2\mu_2 = 0 \\ x \leq y, \quad x + 2y \leq 2 \\ \mu_1 \geq 0, \mu_2 \geq 0 \\ \mu_1 = 0 \quad \text{or} \quad x = y \\ \mu_2 = 0 \quad \text{or} \quad x + 2y = 2 \end{cases}$$

If $\mu_1 = \mu_2 = 0$ we get $x = 1$, $y = 2$ thus $x + 2y = 5 > 2$ not admissible.
If $\mu_1 = 0$ and $\mu_2 > 0$, we get $x = 2 - 2y$ and $\mu_2 = 2(1-x) = 4y-2$, leading to $2(y-2)+2(4y-2) = 0$. Thus, $y = 4/5$, $x = 2/5$, $\mu_1 = 0$, $\mu_2 = 6/5 > 0$ satisfy KKT conditions, and thus is optimal by convexity.

Exercise 2 (First order optimality condition)**.** *Consider, for f differentiable,*

$$\begin{aligned} (P) \quad \underset{x\in\mathbb{R}^n}{\text{Min}} \quad & f(x) \\ s.t. \quad & x \in X \end{aligned}$$

Recall that

$$T_X(x_0) = \Big\{ \ d \in \mathbb{R}^n \mid \exists d_k \to d, \exists t_k \searrow 0, \\ s.t. \ x_0 + t_k d_k \in X \Big\}$$

and $K^\oplus = \big\{\lambda \mid \lambda^\top x \geq 0, \ \forall x \in K\big\}$.
Show that

1. *If x_0 is an optimal solution to (P), then $\nabla f(x_0) \in [T_X(x_0)]^\oplus$.*
2. *If f is convex, X is closed convex, and $\nabla f(x_0) \in [T_X(x_0)]^\oplus$, then x_0 is an optimal solution to (P).*

Answers:

1. Assume that $\nabla f(x_0) \notin [T_X(x_0)]^\oplus$. Then we have $d \in T_X(x_0)$ such that $d^\top \nabla f(x_0) < 0$. By continuity of scalar product we have, for k large enough, $d_k^\top \nabla f(x_0) < 0$. We have $x_0 + t_k d_k \in X$, and $f(x_0 + t_k d_k) = f(x_0) + t_k d_k^\top \nabla f(x_0) + o(t_k d_k)$. Thus, for k large enough, $f(x_0 + t_k d_k) < f(x_0)$.
2. By convexity of X, we have, for $x \in X$, $(x - x_0) \in T_X(x_0)$. Further, by convexity of f, $f(x) \geq f(x_0) + \big\langle \nabla f(x_0) \, , x - x_0 \big\rangle \geq f(x_0)$.

Exercise 3. *In the following cases, are the KKT conditions necessary / sufficient ?*

1.

$$\begin{aligned} \underset{x_1,x_2,x_3}{\min} \quad & 12x_1 - 5x_2 + 3x_3 \\ s.t. \quad & x_1 + 2x_2 - x_3 = 5 \\ & x_1 - x_2 \geq -2 \\ & 2x_1 - 4x_2 \leq 12 \end{aligned}$$

2.

$$\begin{aligned} \min_{x_1,x_2,x_3} \quad & 4x_1^2 - x_1x_2 + x_2^2 - 12x_1 \\ s.t. \quad & x_1 - 2x_2 + x_3 = 5 \\ & x_1^2 + 3x_2^2 \leq 10 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

3.

$$\begin{aligned} \min_{x_1,x_2,x_3} \quad & e^{x_1} - x_1x_2 + x_3^3 \\ s.t. \quad & \ln(e^{x_1-4x_2} + e^{x_1+x_3}) \leq 2x_1 + 3 \\ & 2x_1^2 + x_2^2 \leq 2 \end{aligned}$$

4.

$$\begin{aligned} \min_{x_1,x_2} \quad & -x_1 \\ s.t. \quad & -x_2 - (x_1-1)^3 \leq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

5.

$$\begin{aligned} \min_{x_1,x_2} \quad & -x_1 \\ s.t. \quad & x_2 - (x_1-1)^3 \leq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Answers:

1. CNS as problem is linear, thus convex and qualified everywhere
2. CNS as problem is convex and qualified by Slater
3. CN as constraints are convex and qualified by Slater but objective is nonconvex
4. CN, constraints are qualified due to "positive-independence" condition.
5. Neither. Indeed, no sufficient qualification conditions are satisfied and we can even check that the constraints are not qualified at $x_0 = (1,0)$. Indeed, we have ($x_1 \geq 0$ is not active at x_0)

$$T^\ell_{x_0}X = \{x \mid x_2 - 0 \leq 0,\ x_2 \leq 0\} = \mathbb{R} \times \{0\};$$

$$T_{x_0}X = \mathbb{R}^+ \times \{0\}.$$

Exercise 4. *Solve the following problem using first order optimality conditions*

$$\begin{aligned} \min_{x_1,x_2} \quad & -2(x_1-2)^2 - x_2^2 \\ s.t. \quad & x_1^2 + x_2^2 \leq 25 \\ & x_1 \geq 0 \end{aligned}$$

Answers: First note that the constraint set is convex, and $(1,1)$ is a Slater's point, ensuring qualification everywhere.
The Lagrangian reads

$$\mathcal{L}(x_1,x_2,\mu_1,\mu_2) = -2(x_1-2)^2 - x_2^2 + \mu_1(x_1^2+x_2^2-25) - \mu_2 x_1$$

The KKT conditions thus read

$$\begin{cases} -4(x_1-2) + 2\mu_1 x_1 - \mu_2 = 0 \\ -2x_2 + 2\mu_1 x_2 = 0 \\ x_1^2 + x_2^2 \leq 25 \\ x_1 \geq 0 \\ \mu_1, \mu_2 \geq 0 \\ \mu_1 = 0 \quad \text{or} \quad x_1^2 + x_2^2 = 25 \\ \mu_2 = 0 \quad \text{or} \quad x_1 = 0 \end{cases}$$

If $\mu_1 = \mu_2 = 0$, we have $x_1 = 2$ and $x_2 = 0$ which satisfies the primal constraints. Thus $\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}$ is a primal-dual point satisfying KKT conditions with associated value 0.
If $\mu_1 = 0$ and $\mu_2 > 0$ we have $x_1 = x_2 = 0$ with $\mu_2 = 8 > 0$ which is a primal-dual point with value -8.
If $\mu_2 = 0$ and $\mu_1 > 0$ we have

$$\begin{cases} -4(x_1-2) + 2\mu_1 x_1 = 0 \\ -2x_2 + 2\mu_1 x_2 = 0 \\ x_1 \geq 0 \\ \mu_1 > 0 \\ x_1^2 + x_2^2 = 25 \end{cases}$$

Thus, either $x_2 = 0$ or $\mu_1 = 1$. In the first case we get $x_1 = 5$, $x_2 = 0$, thus $\mu_1 = 6/5 > 0$ and $\mu_2 = 0$ which is a KKT point with value -18. In the second case we get $x_1 = 4$ and $x_2 = \pm 3$, with $\mu_1 = 1$ and $\mu_2 = 0$ which are two KKT points with value -17.
Finally, if $\mu_2 > 0$ and $\mu_1 > 0$, we have $x_1 = 0$ and $x_2 = \pm 5$ with $\mu_1 = 1$ and $\mu_2 = 8$, which are two

KKT points with value -33, and thus the global minima.

Exercise 5 (When KKT can fail without qualification)**.** *Consider the problem*

$$\min_{x\in\mathbb{R}} \; f(x) := x \qquad s.t. \qquad g(x) := x^2 \le 0.$$

1. *Find the (global) solution.*
2. *Write the KKT conditions at the solution and show that there is* no *Lagrange multiplier* $\mu \ge 0$ *satisfying them.*
3. *Compute the tangent cone* $T_X(x^\star)$ *and the linearized cone* $T_X^\ell(x^\star)$ *at the solution* $x^\star$ *(with* $X = \{x : \; g(x) \le 0\}$*).*

Answers:

1. Since $x^2 \le 0$ iff $x = 0$, the feasible set is $X = \{0\}$, hence the unique feasible point
$$x^\star = 0$$
is the (global) minimizer.

2. The Lagrangian is $\mathcal{L}(x, \mu) = x + \mu x^2$ with $\mu \ge 0$. The KKT conditions at $x^\star = 0$ are:
$$g(x^\star) \le 0, \quad \mu \ge 0, \quad \mu g(x^\star) = 0, \quad \nabla f(x^\star) + \mu \nabla g(x^\star) = 0.$$
Here $\nabla f(0) = 1$ and $\nabla g(0) = 2x\big|_{x=0} = 0$, so stationarity becomes
$$1 + \mu \cdot 0 = 0,$$
which is impossible. Therefore, *no* multiplier $\mu \ge 0$ satisfies KKT at the (global) solution.

3. Since $X = \{0\}$, any feasible sequence is constant, hence the tangent cone is
$$T_X(0) = \{0\}.$$
The linearized cone for the inequality $g(x) \le 0$ at 0 is
$$T_X^\ell(0) = \{d \in \mathbb{R} : \; \nabla g(0)\, d \le 0\} = \{d \in \mathbb{R} : \; 0 \le 0\} = \mathbb{R}.$$
Thus $T_X^\ell(0) \ne T_X(0)$.