# Exercises: Constrained Optimization

**Exercise 1** (Penalization)**.** *We consider the following problem*

$$(P) \quad \min_{x \in \mathbb{R}^n} \quad f(x) \qquad s.t. \quad Ax = b, \quad x \leq 0$$

*with value $v$ and the following penalized versions*

$$(P_t^{in}) \quad \min_{x \in \mathbb{R}^n} f(x) - t\sum_{i=1}^n \ln(-x_i) \qquad s.t. Ax = b, \quad x < 0$$

*and*

$$(P_t^{out}) \quad \min_{x \in \mathbb{R}^n} f(x) + t\sum_{i=1}^n (x_i)^+ \qquad s.t. \quad Ax = b$$

*with associated value $v_t^{in}$ and $v_t^{out}$, and an optimal solution $x_t^{in}$ and $x_t^{out}$.*

1. *Intuitively, assuming that $f$ is "well behaved", for $t$ going to which value does $(P_t^{in})$ tends to the original problem $(P)$ ? In which sense ?*
2. *What can you say about $x_t^{in}$ ?*
3. *Can you compare $v_t^{in}$ and $v$ ?*
4. *Same questions for $(P_t^{out})$.*

**Exercise 2** (Decomposition by prices)**.** *We consider the following energy problem:*

- *you are an energy producer with $N$ production units*
- *you have to satisfy a given demand planning for the next 24h (i.e. the total output at time $t$ should be equal to $d_t$)*
- *the time step is the hour, and each unit have a production cost for each planning given as a convex quadratic function of the planning*
- *For each unit $i$, the production planning $u^i = (u_t^i)_{t \in [24]}$ has to satisfy polyhedral constraints $u^i \in U^i$.*

1. *Model this problem as an optimization problem. In which class does it belongs ? How many variables ?*
2. *Apply Uzawa's algorithm to this problem. Why could this be an interesting idea ?*
3. *Give an economic interpretation to this method.*
4. *What would happen if each unit had production constraints ?*

**Exercise 3** (Kelley's convergence)**.** *We are going to prove that, if $f : \mathbb{R}^n \to \mathbb{R}$ is convex, and $X$ a non-empty polytope (bounded polyhedron) then Kelley's cutting plane algorithm is converging. Consider $x_1 \in X$. We consider a sequence of points $(x^{(k)})_{k \in \mathbb{N}}$ such that $x^{(k+1)}$ is an optimal solution to*

$$(\mathcal{P}^{(k)}) \; \underline{v}^{(k+1)} = \operatorname*{Min}_{x \in X} z \qquad s.t. \; f(x^{(\kappa)}) + \left\langle g^{(\kappa)}, x - x^{(\kappa)} \right\rangle \leq z \quad \forall \kappa \in [k]$$

*where $g^{(k)} \in \partial f(x^{(k)})$.*
*Denote $v = \min_{x \in X} f(x)$.*

1. *Show that $v$ exists and is finite, and that there exists a sequence $x^{(k)}$.*
2. *Show that there exists $L$ such that, for all $k_1$ and $k_2$, we have $\|f(x^{(k_1)}) - f(x^{(k_2)})\| \leq L\|x^{(k_1)} - x^{(k_2)}\|$, and $\|g^{(k)}\| \leq L$.*

3. *Let* $K_\varepsilon = \{k \in \mathbb{N} \mid f(x^{(k)}) > v + \varepsilon\}$ *be the set of index such that* $x^{(k)}$ *is not an* $\varepsilon$*-optimal solution. Show that* $f(x_k) \to v$ *if and only if* $K_\varepsilon$ *is finite for all* $\varepsilon > 0$

4. *Consider* $k_1, k_2 \in K_\varepsilon$*, such that* $k_2 > k_1$*. Show that*

$$f(x^{(k_1)}) + \left\langle g^{(k_1)}, x^{(k_2)} - x^{(k_1)} \right\rangle \leq \underline{v}^{(k_2)} \leq v$$

5. *Show that* $\varepsilon + f(x^{(k_1)}) + \left\langle g^{(k_1)}, x^{(k_2)} - x^{(k_1)} \right\rangle < f(x^{(k_2)})$

6. *Show that* $\varepsilon < 2L\|x^{(k_2)} - x^{(k_1)}\|$.

7. *Prove that* $f(x^{(k)}) \to v$.

8. *(Optional - hard) Find a complexity bound for the method (that is a number of iteration* $N_\varepsilon$ *after which you are sure to have obtained a* $\varepsilon$*-optimal solution).*