Exercises : optimization problem classes

Exercise 1 (Hyperbolic constraints as SOCP).

1. *Show that, for all* $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $z \in \mathbb{R}$,

$$x^\top x \le yz, \quad y \ge 0, \quad z \ge 0$$

iff

$$\left\| \begin{pmatrix} 2x \\ y - z \end{pmatrix} \right\| \le y + z \quad y \ge 0, \quad z \ge 0$$

2. *Represent the following problem as an SOCP*

$$\begin{aligned} (P) \ \text{Max} \qquad & \left(\sum_{i=1}^{n} 1/(a_i^\top x - b) \right)^{-1} \\ s.t. \qquad & Ax > b \end{aligned}$$

Exercise 2. *We consider a physical function* Φ *that is approximated as the superposition of multiple simple phenomenon (e.g. waves). Each simple phenomenon* $p \in [P]$ *is represented by a function* $\Phi_p : \mathbb{R}^d \to \mathbb{R}$.
We have data points $(x^k, y^k)_{k \in [n]}$, *and want to find the* Φ *that match* at best *the data while being a linear combination of* Φ_p.
Propose a least-square regression that answer this question.

Exercise 3. *Consider a chocolate manufacturing company that produces only two types of chocolate – A and B. Both the chocolates require Milk and Choco only. To manufacture each unit of A and B, the following quantities are required:*

- *Each unit of A requires 1 unit of Milk and 3 units of Choco*
- *Each unit of B requires 1 unit of Milk and 2 units of Choco*

The company kitchen has a total of 5 units of Milk and 12 units of Choco. On each sale, the company makes a profit of

- *6 per unit A sold*
- *5 per unit B sold.*

Model this as an LP.

Exercise 4. *A classical extension of the least-square problem, which has strong theoretical and practical intereset is the LASSO problem*

$$\underset{x \in \mathbb{R}^p}{\text{Min}} \qquad \|Ax - b\|^2 + \lambda \|x\|_1$$

Show that this problem can be cast as a QP problem.

Exercise 5. *Consider the following optimization problem.*

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{Min}} \qquad & c^\top x \\ s.t. \qquad & Ax = b \\ & x_i \in \{0, 1\} \qquad \forall i \in I \end{aligned}$$

Write this problem as a QCQP. Is it convex ?

Exercise 6. *Consider a facility that plan to deliver product to clients by drone (thus in direct line). Assume that you have N clients, each with position (in* $\mathbb{R}^2$*)* x_n. *The drone make each time a direct travel from the facility location to the client. Assume that the drone have a maximum range of R, and that you want to minimize the average travel distance while being able to serve all of your clients.*
Model the problem of choosing the facility location as an SOCP.

Exercise 7. *Consider the following robust linear program*

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{Min}} \qquad & c^\top x \\ s.t. \qquad & (a_i + R_i \delta_i)^\top x \le b_i \quad \forall \|\delta_i\|_2 \le 1, \quad \forall i \in [m] \end{aligned}$$

where R_i *are positive real numbers. Write this problem as an SOCP.*

Exercise 8. *Let $F(\theta)$ be a symmetric matrix parametrized by $\theta \in \mathbb{R}^d$ whose coefficients are linear in θ. Model the problem of finding the parameter $\theta \in \Theta$, where Θ is a polyhedron, minimizing $\kappa(\theta)$ as an SDP.*
What happen if the coefficient of $F(\theta)$ are affine in θ ? Suggest a solution method ? (hard)

Exercise 9. *Consider a finite set $X = \{x_i\}_{i\in[n]}$, and $\mathcal{P}^+$ the set of probabilities on X. For $\mathbb{P}, \mathbb{Q} \in \mathcal{P}$, with $\mathrm{supp}(\mathbb{Q}) = X$, we define the* Kullback-Leibler *divergence as*

$$d_{KL}(\mathbb{P}|\mathbb{Q}) = \sum_{i=1}^{n} p_i \ln(p_i/q_i)$$

where $p_i = \mathbb{P}(X = x_i)$ and $q_i = \mathbb{Q}(X = x_i)$.
Let X be 100 equidistant points spanning in $[-1, 1]$. Let $\mathbb{Q}$ be uniform on X.
We are looking for the probability $\mathbb{P}$ on X such that

- $\mathbb{E}_{\mathbb{P}}[\boldsymbol{X}] \in [-0.1, 0.1]$
- $\mathbb{E}_{\mathbb{P}}[\boldsymbol{X}^2] \in [0.5, 0.6]$
- $\mathbb{E}_{\mathbb{P}}[3\boldsymbol{X}^2 - 2\boldsymbol{X}] \in [-0.3, -0.2]$
- $\mathbb{P}(\boldsymbol{X} < 0) \in [0.3, 0.4]$

that minimize the Kullback-Leibler divergence from $\mathbb{Q}$.
Model this problem as an optimization problem. In which class does it belongs ?

Exercise 10. *Consider that you sell a given product over T days. The demand for each day is d_t. Having a quantity x_t of items in stock have a cost (per day) of cx_t. You can order, each day, a quantity q_t, and have to satisfy the demand.*
For each of the following variation : model the problem, explicit the class to which it belongs, and give the optimal solution if easily found.

1. *Without any further constraint / specifications.*
2. *There is an "ordering cost": each time you order, you have to pay a fix cost κ.*
3. *Instead of an "ordering cost" there is a maximum number of days at which you can order a replenishment.*