# Exercises : optimization problem classes

**Exercise 1** (Hyperbolic constraints as SOCP)**.**

1. Show that, for all $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $z \in \mathbb{R}$,

$$x^\top x \leq yz, \quad y \geq 0, \quad z \geq 0$$

   *iff*

$$\left\| \begin{pmatrix} 2x \\ y - z \end{pmatrix} \right\| \leq y + z \quad y \geq 0, \quad z \geq 0$$

2. *Represent the following problem as an SOCP*

$$(P) \text{ Max} \quad \left( \sum_{i=1}^{n} 1/(a_i^\top x - b) \right)^{-1}$$
$$\text{s.t.} \quad Ax > b$$

**Answers:**

1. Assume $y \geq 0, z \geq 0$, then $y + z \geq 0 \iff xy \geq 0$. We now assume $y + z \geq 0$, then we have

$$\left\| \begin{pmatrix} 2x \\ y - z \end{pmatrix} \right\| \leq y + z$$
$$\iff \left\| \begin{pmatrix} 2x \\ y - z \end{pmatrix} \right\|^2 \leq (y + z)^2$$
$$\iff 4x^\top x + (y - z)^2 \leq y^2 + 2yz + z^2$$
$$\iff 4x^\top x \leq 4yz$$
$$\iff x^\top x \leq yz$$

2. Since $t \mapsto 1/t$ is decreasing $(P)$ is equivalent to

$$\text{Min} \quad \sum_{i=1}^{n} 1/(a_i^\top x - b)$$
$$\text{s.t.} \quad Ax > b$$

By adding the lift variables $z_i$, $(P)$ is equivalent to the problem

$$\text{Min} \quad \sum_{i=1}^{n} z_i$$
$$\text{s.t.} \quad Ax > b$$
$$1/(a_i^\top x - b) \geq z_i \geq 0, \forall i \in [n]$$

which is equivalent to

$$\text{Min} \quad \sum_{i=1}^{n} z_i$$
$$\text{s.t.} \quad Ax \geq b$$
$$1 \leq z(a_i^\top x - b), \forall i \in [n]$$
$$z_i \geq 0, \forall i \in [n]$$

By question 1. it is equivalent to the following SOCP:

$$\text{Min} \quad \sum_{i=1}^{n} z_i$$
$$\text{s.t.} \quad - Ax \leq -b$$
$$\left\| \begin{pmatrix} 2 \\ a_i^\top x - b - z \end{pmatrix} \right\| \leq a_i^\top x - b + z, \forall i \in [n]$$
$$z_i \geq 0, \forall i \in [n]$$

**Exercise 2.** *We consider a physical function $\Phi$ that is approximated as the superposition of multiple simple phenomenon (e.g. waves). Each simple phenomenon $p \in [P]$ is represented by a function $\Phi_p : \mathbb{R}^d \to \mathbb{R}$.*
*We have data points $(x^k, y^k)_{k \in [n]}$, and want to find the $\Phi$ that match at best the data while being a linear combination of $\Phi_p$.*
*Propose a least-square regression that answer this question.*

**Answers:** We define the matrix $M \in \mathbb{R}^{n \times P}$ with coefficients $M_{k,p} = \Phi_p(x_k)$. We propose the following last square regression problem:

$$\underset{\alpha \in \mathbb{R}^P}{\text{Min}} \quad \|M\alpha - y\|^2 + \lambda \|\alpha\|_1$$

**Exercise 3.** *Consider a chocolate manufacturing company that produces only two types of chocolate – A and B. Both the chocolates require Milk and Choco only. To manufacture each unit of A and B, the following quantities are required:*

- *Each unit of A requires 1 unit of Milk and 3 units of Choco*

- *Each unit of B requires 1 unit of Milk and 2 units of Choco*

*The company kitchen has a total of 5 units of Milk and 12 units of Choco. On each sale, the company makes a profit of*

- *6 per unit A sold*

- *5 per unit B sold.*

*Model this as an LP.*

**Exercise 4.** *A classical extension of the least-square problem, which has strong theoretical and practical intereset is the LASSO problem*

$$\underset{x \in \mathbb{R}^p}{\text{Min}} \quad \|Ax - b\|^2 + \lambda \|x\|_1$$

*Show that this problem can be cast as a QP problem.*

**Answers:** The LASSO problem is equivalent to the following QP problem

$$\underset{x \in \mathbb{R}^n, z \in \mathbb{R}^n}{\text{Min}} \quad x^\top A^\top A x - 2b^\top A x + \lambda \sum_{i=1}^{n} z_i$$

$$\text{s.t.} \quad x_i \le z_i$$
$$\quad - x_i \le z_i$$

**Exercise 5.** *Consider the following optimization problem.*

$$\underset{x \in \mathbb{R}^n}{\text{Min}} \quad c^\top x$$
$$\text{s.t.} \quad Ax = b$$
$$\quad x_i \in \{0, 1\} \qquad \forall i \in I$$

*Write this problem as a QCQP. Is it convex ?*

**Answers:** The constraint $x_i \in \{0, 1\}$ is equivalent to $x_i(1 - x_i) = 0$. We define $q_i := (0, \cdots, 0, 1, 0, \cdots, 0)$ as the vector with all coordinates equal to 0 except the ith which equals 1. We set $Q_i = 2\,diag(q_i) = 2q_i q_i^\top$. Then, this problem is equivalent to the following QCQP problem

$$\underset{x \in \mathbb{R}^n}{\text{Min}} \quad c^\top x$$
$$\text{s.t.} \quad Ax = b$$
$$\quad \frac{1}{2}x^\top Q_i x + q_i^\top x \le 0 \qquad \forall i \in I$$
$$\quad - \frac{1}{2}x^\top Q_i x - q_i^\top x \le 0 \qquad \forall i \in I$$

It is not convex in the general case (i.e. if the admissible set is neither empty or reduced to a singleton) since the set $\{0,1\}^n$ is not convex. Remark that $-Q_i$ is not positive.

**Exercise 6.** *Consider a facility that plan to deliver product to clients by drone (thus in direct line). Assume that you have N clients, each with position (in $\mathbb{R}^2$) $x_n$. The drone make each time a direct travel from the facility location to the client. Assume that the drone have a maximum range of R, and that you want to minimize the average travel distance while being able to serve all of your clients.*
*Model the problem of choosing the facility location as an SOCP.*

**Answers:** We want to minimize the average travel distance $\frac{1}{N}\sum_{n=1}^{N} \|x_n - y\|$ from a center $y$ to the clients $(x_n)$ while being able to serve all of your clients. We modelize this by the problem

$$\underset{y \in \mathbb{R}^2}{\text{Min}} \quad \frac{1}{N}\sum_{n=1}^{N} \|x_n - y\| \tag{1}$$

$$\text{s.t.} \quad \|x_n - y\| \le R, \forall n \in [N] \tag{2}$$

By adding lift variables $z_n$, this equivalent to the following SOCP:

$$\underset{y \in \mathbb{R}^2, z \in \mathbb{R}^N}{\text{Min}} \quad \frac{1}{n}\sum_{n=1}^{N} z_n \tag{3}$$

$$\text{s.t.} \quad \|x_n - y\| \le R, \forall n \in [N] \tag{4}$$
$$\quad \|x_n - y\| \le z_n, \forall n \in [N] \tag{5}$$

**Exercise 7.** *Consider the following robust linear program*

$$\underset{x\in\mathbb{R}^n}{\text{Min}} \quad c^\top x$$

$$s.t. \quad (a_i + R_i\delta_i)^\top x \le b_i \quad \forall \|\delta_i\|_2 \le 1, \quad \forall i \in [m]$$

*where $R_i$ are positive real numbers. Write this problem as an SOCP.*

**Answers:** The constraint $(a_i + R_i\delta_i)^\top x \le b_i, \quad \forall\|\delta_i\|_2 \le 1$ is equivalent to

$$\max_{\delta_i \mid \|\delta_i\|_2 \le 1} (a_i + R_i\delta_i)^\top x \le b_i$$

which is equivalent to

$$R_i \max_{\delta_i \mid \|\delta_i\|_2 \le 1} \delta_i^\top x \le b_i - a_i^\top x.$$

However, $\max_{\delta_i \mid \|\delta_i\|_2 \le 1} \delta_i^\top x = \|x\|$. Indeed, this result is trivial for $x = 0$, for $x \ne 0$, we have that $\|x\|$ is an upper bound by Cauchy-Swhartz inequality which is attained for $\delta_i = \frac{x}{\|x\|}$. Thus, our problem is equivalent to the following SOCP:

$$\underset{x\in\mathbb{R}^n}{\text{Min}} \quad c^\top x$$

$$s.t. \quad R_i\|x\| \le -a_i^\top x + b_i, \quad \forall i \in [m]$$

**Exercise 8.** *Let $F(\theta)$ be a symmetric matrix parametrized by $\theta \in \mathbb{R}^d$ whose coefficients are linear in $\theta$. Model the problem of finding the parameter $\theta \in \Theta$, where $\Theta$ is a polyhedron, minimizing $\kappa(\theta)$ as an SDP.*
*What happen if the coefficient of $F(\theta)$ are affine in $\theta$ ? Suggest a solution method ? (hard)*

**Exercise 9.** *Consider a finite set $X = \{x_i\}_{i\in[n]}$, and $\mathcal{P}^+$ the set of probabilities on $X$. For $\mathbb{P}, \mathbb{Q} \in \mathcal{P}$, with $\text{supp}(\mathbb{Q}) = X$, we define the Kullback-Leibler divergence as*

$$d_{KL}(\mathbb{P}|\mathbb{Q}) = \sum_{i=1}^{n} p_i \ln(p_i/q_i)$$

*where $p_i = \mathbb{P}(X = x_i)$ and $q_i = \mathbb{Q}(X = x_i)$.*
*Let $X$ be 100 equidistant points spanning in $[-1, 1]$. Let $\mathbb{Q}$ be uniform on $X$.*
*We are looking for the probability $\mathbb{P}$ on $X$ such that*

- $\mathbb{E}_{\mathbb{P}}[X] \in [-0.1, 0.1]$
- $\mathbb{E}_{\mathbb{P}}[X^2] \in [0.5, 0.6]$
- $\mathbb{E}_{\mathbb{P}}[3X^2 - 2X] \in [-0.3, -0.2]$
- $\mathbb{P}(X < 0) \in [0.3, 0.4]$

*that minimize the Kullback-Leibler divergence from $\mathbb{Q}$.*
*Model this problem as an optimization problem. In which class does it belongs ?*

**Exercise 10.** *Consider that you sell a given product over $T$ days. The demand for each day is $d_t$. Having a quantity $x_t$ of items in stock have a cost (per day) of $cx_t$. You can order, each day, a quantity $q_t$, and have to satisfy the demand.*
*For each of the following variation : model the problem, explicit the class to which it belongs, and give the optimal solution if easily found.*

1. *Without any further constraint / specifications.*

2. *There is an "ordering cost": each time you order, you have to pay a fix cost $\kappa$.*

3. *Instead of an "ordering cost" there is a maximum number of days at which you can order a replenishment.*