Exercises: Duality

April 12, 2023

Exercise 1 (Dual formulation)**.** *Let $g : \mathbb{R}^n \to \mathbb{R}^m$. Show that*

1. $\mathbb{I}_{g(x)=0} = \sup_{\lambda \in \mathbb{R}^m} \lambda^\top g(x)$
2. $\mathbb{I}_{g(x)\leq 0} = \sup_{\lambda \in \mathbb{R}^m_+} \lambda^\top g(x)$
3. $\mathbb{I}_{g(x)\in C} = \sup_{\lambda \in -C^\oplus} \lambda^\top g(x)$ *where C is a closed convex cone, and $C^\oplus := \{\lambda \in \mathbb{R}^m \mid \lambda^\top c \geq 0, \forall c \in C\}$.*

Exercise 2 (Linear Programming)**.** *Consider the following linear problem (LP)*

$$
\begin{aligned}
(P) \quad \underset{x\geq 0}{\text{Min}} \quad & c^\top x \\
s.t. \quad & Ax = b
\end{aligned}
$$

1. *Show that the dual of (P) is an LP.*
2. *Show that the dual of the dual of (P) is equivalent to (P).*

Exercise 3 (Quadratically Constrained Quadratic Programming)**.** *Consider the problem*

$$
\begin{aligned}
(QCQP) \quad \underset{x\in\mathbb{R}^n}{\text{Min}} \quad & \frac{1}{2}x^\top P_0 x + q_0^\top x + r_0 \\
& \frac{1}{2}x^\top P_i x + q_i^\top x + r_i \leq 0 \quad \forall i \in [m]
\end{aligned}
$$

where $P_0 \in S^n_{++}$ and $P_i \in S^n_+$.

1. *Show by duality that, for $\mu \in \mathbb{R}^m_+$, there exists P_μ, q_μ and r_μ,such that $g(\mu) = -\frac{1}{2}q_\mu P_\mu^{-1} q_\mu + r_\mu \leq val(P)$.*
2. *Give an easy condition under which $val(P) = \sup_{\mu\geq 0} g(\mu)$.*

Exercise 4 (Conic Programming)**.** *Let $K \subset \mathbb{R}^n$ be a closed convex pointed cone, and denote $x \preceq_K y$ iff $y \in x + K$. Consider the following program, with $A \in M_{m,n}$ and $b \in \mathbb{R}^m$.*

$$
\begin{aligned}
(P) \quad \underset{x\in\mathbb{R}^n}{\text{Min}} \quad & c^\top x \\
s.t. \quad & Ax = b \\
& x \preceq_K 0
\end{aligned}
$$

1. *Show that (P) is a convex optimization problem.*
2. *Denote $\mathcal{L}(x,\lambda,\mu) = c^\top x + \lambda^\top(Ax - b) + \mu^\top x$. Show that $\text{val}(P) = \text{Min}_{x\in\mathbb{R}^n} \sup_{\lambda\in\mathbb{R}^m, \mu\in K^\oplus} \mathcal{L}(x,\lambda,\mu)$.*
3. *Give a dual problem to (P).*

Exercise 5 (Duality gap)**.** *Consider the following problem*

$$
\begin{aligned}
\underset{x\in\mathbb{R}, y\in\mathbb{R}^+_*}{\text{Min}} \quad & e^{-x} \\
s.t. \quad & x^2/y \leq 0
\end{aligned}
$$

1. *Find the optimal solution of this problem.*
2. *Write and solve the (Lagrangian) dual problem. Is there a duality gap ?*

Exercise 6 (Two-way partitionning)**.** *Let $W \in S_n$ be a symmetric matrix, consider the following problem.*

$$
\begin{aligned}
(P) \quad \underset{x\in\mathbb{R}^n}{\text{Min}} \quad & x^\top W x \\
s.t. \quad & x_i^2 = 1 \qquad \forall i \in [n]
\end{aligned}
$$

1. *Consider a set of n element that you want to partition in 2 subsets, with a cost $c_{i,j}$ if i and j are in the same set, and a cost $-c_{i,j}$ if they are in a different set. Justify that it can be solved by solving (P).*

2. *Is (P) a convex problem ?*

3. *Show that, for any $\lambda \in \mathbb{R}^n$ such that $W + \operatorname{diag}(\lambda) \succeq 0$, we have $\operatorname{val}(P) \geq -\sum \lambda_i$. Deduce a lower bound on $\operatorname{val}(P)$.*

Exercise 7 (Linear SVM : duality)**.** *Consider the following problem (see : `https://www.youtube.com/watch?v=IOetFPgsMUc` for background)*

$$
\begin{aligned}
\min_{w\in\mathbb{R}^d, b\in\mathbb{R}} \ & \frac{1}{2}\|w\|^2 \\
s.t. \quad & y_i(w^\top x_i + b) \geq 1 && \forall i \in [n] \\
& \eta_i \geq 0 && \forall i \in [n]
\end{aligned}
$$

1. *In which case can we guarantee strong duality ?*

2. *Write the dual of this optimization problem and express the optimal primal solution $(w^\sharp, b^\sharp)$ in terms of the optimal dual solution.*

Exercise 8. *We consider the following problem.*

$$
\begin{aligned}
\operatorname*{Min}_{x_1,x_2} \quad & x_1^2 + x_2^2 && (1) \\
s.t. \quad & (x_1-1)^2 + (x_2-1)^2 \leq 1 && (2) \\
& (x_1-1)^2 + (x_2+1)^2 \leq 1 && (3)
\end{aligned}
$$

1. *Classify this problem (After 5th course)*

2. *Find the optimal solution and value of this problem.*

3. *Write and solve the KKT equation for this problem.*

4. *Derive and solve the Lagrangian dual of this problem.*

5. *Do we have strong duality ? If yes, could we have known it from the start ? If not, can you comment on why ?*