Exercises : Optimality conditions

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Exercise 1. *Solve the following optimization problem*

$$\begin{aligned} \operatorname*{Min}_{x,y\in\mathbb{R}^2} \quad & (x-1)^2+(y-2)^2 \\ & x\le y \\ & x+2y\le 2 \end{aligned}$$

Answers: The problem is convex and qualified through Slater's condition (e.g. $(-1,0)$). Lagrangian

$$\begin{aligned}\mathcal{L}(x,y,\mu) =& (x-1)^2+(y-2)^2 \\ &+\mu_1(x-y)+\mu_2(x+2y-2)\end{aligned}$$

KKT conditions

$$\begin{cases} 2(x-1)+\mu_1+\mu_2=0 \\ 2(y-2)-\mu_1+2\mu_2=0 \\ x\le y, \quad x+2y\le 2 \\ \mu_1\ge 0, \mu_2\ge 0 \\ \mu_1=0 \quad \text{or} \quad x=y \\ \mu_2=0 \quad \text{or} \quad x+2y=2 \end{cases}$$

If $\mu_1=\mu_2=0$ we get $x=1$, $y=2$ thus $x+2y=5>2$ not admissible.
If $\mu_1=0$ and $\mu_2>0$, we get $x=2-2y$ and $\mu_2=2(1-x)=4y-2$, leading to $2(y-2)+2(4y-2)=0$. Thus, $y=4/5$, $x=2/5$, $\mu_1=0$, $\mu_2=6/5>0$ satisfy KKT conditions, and thus is optimal by convexity.

Exercise 2 (First order optimality condition)**.** *Consider, for f differentiable,*

$$\begin{aligned} (P) \quad \operatorname*{Min}_{x\in\mathbb{R}^n} \quad & f(x) \\ s.t. \quad & x\in X \end{aligned}$$

Recall that

$$\begin{aligned} T_X(x_0)=\big\{\ d\in\mathbb{R}^n \mid \exists d_k\to d, \exists t_k\searrow 0, \\ s.t.\ x_0+t_kd_k\in X\big\} \end{aligned}$$

and $K^\oplus=\big\{\lambda \mid \lambda^\top x\ge 0,\ \forall x\in K\big\}$.
Show that

1. *If x_0 is an optimal solution to (P), then $\nabla f(x_0)\in[T_X(x_0)]^+$.*
2. *If f is convex, X is closed convex, and $\nabla f(x_0)\in[T_X(x_0)]^+$, then x_0 is an optimal solution to (P).*

Answers:

1. Assume that $\nabla f(x_0)\notin[T_X(x_0)]^+$. Then we have $d\in T_X(x_0)$ such that $d^\top\nabla f(x_0)<0$. By continuity of scalar product we have, for k large enough, $d_k^\top\nabla f(x_0)<0$. We have $x_0+t_kd_k\in X$, and $f(x_0+t_kd_k)=f(x_0)+t_kd_k^\top\nabla f(x_0)+o(t_kd_k)$. Thus, for k large enough, $f(x_0+t_kd_k)<f(x_0)$.
2. By convexity of X, we have, for $x\in X$, $(x-x_0)\in T_X(x_0)$. Further, by convexity of f, $f(x)\ge f(x_0)+\big\langle\nabla f(x_0)\,,x-x_0\big\rangle\ge f(x_0)$.

Exercise 3. *In the following cases, are the KKT conditions necessary / sufficient ?*

1.

$$\begin{aligned} \min_{x_1,x_2,x_3} \quad & 12x_1-5x_2+3x_3 \\ s.t. \quad & x_1+2x_2-x_3=5 \\ & x_1-x_2\ge -2 \\ & 2x_1-4x_2\le 12 \end{aligned}$$

2.

$$\begin{aligned}\min_{x_1,x_2} \quad & 4x_1^2 - x_1x_2 + x_2^2 - 12x_1\\ s.t. \quad & x_1 - 2x_2 + x_3 = 5\\ & x_1^2 + 3x_2^2 \leq 10\\ & x_1, x_2, x_3 \geq 0\end{aligned}$$

3.

$$\begin{aligned}\min_{x_1,x_2,x_3} \quad & e_1^x - x_1x_2 + x_3^3\\ s.t. \quad & \ln(e^{x_1-4x_2} + e^{x_1+x_3}) \leq 2x_1 + 3\\ & 2x_1^2 + x_2^2 \leq 2\end{aligned}$$

4.

$$\begin{aligned}\min_{x_1,x_2} \quad & -x_1\\ s.t. \quad & -x_2 - (x_1-1)^3 \leq 0\\ & x_1, x_2 \geq 0\end{aligned}$$

5.

$$\begin{aligned}\min_{x_1,x_2} \quad & -x_1\\ s.t. \quad & x_2 - (x_1-1)^3 \leq 0\\ & x_1, x_2 \geq 0\end{aligned}$$

Answers:

1. CNS as problem is linear, thus convex and qualified everywhere
2. CNS as problem is convex and qualified by Slater
3. CN as constraints are convex and qualified by Slater but objective is nonconvex
4. CN, constraints are qualified due to "positive-independence" condition.
5. Neither. Indeed, no sufficient qualification conditions are satisfied and we can even check that the constraints are not qualified at $x_0 = (1,0)$. Indeed, we have ($x_1 \geq 0$ is not active at x_0)

$$T^{\ell}_{x_0}X = \{x \mid x_2 - 0 \leq 0,\ x_2 \leq 0\} = \mathbb{R}\times\{0\};$$

$$T_{x_0}X = \mathbb{R}^+ \times \{0\}.$$

Exercise 4. *Solve the following problem using first order optimality conditions*

$$\begin{aligned}\min_{x_1,x_2} \quad & -2(x_1-2)^2 - x_2^2\\ s.t. \quad & x_1^2 + x_2^2 \leq 25\\ & x_1 \geq 0\end{aligned}$$

Answers: First note that the constraint set is convex, and $(1,1)$ is a Slater's point, ensuring qualification everywhere.
The Lagrangian reads

$$\mathcal{L}(x_1,x_2,\mu_1,\mu_2) = -2(x_1-2)^2 - x_2^2 + \mu_1(x_1^2+x_2^2-25) - \mu_2 x_1$$

The KKT conditions thus read

$$\begin{cases}-4(x_1-2) + 2\mu_1 x_1 - \mu_2 = 0\\ -2x_2 + 2\mu_1 x_2 = 0\\ x_1^2 + x_2^2 \leq 25\\ x_1 \geq 0\\ \mu_1, \mu_2 \geq 0\\ \mu_1 = 0 \quad \text{or} \quad x_1^2 + x_2^2 = 25\\ \mu_2 = 0 \quad \text{or} \quad x_1 = 0\end{cases}$$

If $\mu_1 = \mu_2 = 0$, we have $x_1 = 2$ and $x_2 = 0$ which satisfies the primal constraints. Thus $\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}$ is a primal-dual point satisfying KKT conditions with associated value 0.
If $\mu_1 = 0$ and $\mu_2 > 0$ we have $x_1 = x_2 = 0$ with $\mu_2 = 8 > 0$ which is a primal-dual point with value -8.
If $\mu_2 = 0$ and $\mu_1 > 0$ we have

$$\begin{cases}-4(x_1-2) + 2\mu_1 x_1 = 0\\ -2x_2 + 2\mu_1 x_2 = 0\\ x_1 \geq 0\\ \mu_1 > 0\\ x_1^2 + x_2^2 = 25\end{cases}$$

Thus, either $x_2 = 0$ or $\mu_1 = 1$. In the first case we get $x_1 = 5$, $x_2 = 0$, thus $\mu_1 = 6/5 > 0$ and $\mu_2 = 0$ which is a KKT point with value -18. In the second case we get $x_1 = 4$ and $x_2 = \pm 3$, with $\mu_1 = 1$ and $\mu_2 = 0$ which are two KKT points with value -17.
Finally, if $\mu_2 > 0$ and $\mu_1 > 0$, we have $x_1 = 0$ and $x_2 = \pm 5$ with $\mu_1 = 1$ and $\mu_2 = 8$, which are two KKT points with value -33, and thus the global minima.