Exercises : Convex analysis

March 30, 2023

Convex sets

Exercise 1 (Perspective function). *Let $P : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be the* perspective function *defined as $P(x,t) = x/t$, with $\mathrm{dom}(P) = \mathbb{R}^n \times \mathbb{R}_+^*$.*

1. *Show that the image by P of the segment $[\binom{x}{s}, \binom{y}{t}]$ is the segment $[P(\binom{x}{s}), P(\binom{y}{t})]$, i.e. $P([\binom{x}{s}, \binom{y}{t}]) = [P(\binom{x}{s}), P(\binom{y}{t})]$.*
2. *Show that, if $C \subset \mathbb{R}^n \times \mathbb{R}_+^*$ is convex, then $P(C)$ is convex.*
3. *Show that, if $D \subset \mathbb{R}^n$, then $P^{-1}(D)$ is convex.*

Exercise 2 (Dual cones). *Recall that, for any set $K \subset \mathbb{R}^n$, $K^\oplus := \{y \in \mathbb{R}^n \mid \forall x \in K,\ \langle y, x\rangle \geq 0\}$. We say that K is self dual if $K^\oplus = K$.*

1. *Show that $K = \mathbb{R}_+^n$ is self dual.*
2. *We consider the set of symmetric matrices S_n with the scalar product $\langle A, B\rangle = \mathrm{tr}(AB)$. Show that $K = S_n^+(\mathbb{R})$ is self dual.*
3. *Let $\|\cdot\|$ be a norm, show that $K = \{(x,t) \mid \|x\| \leq t\}$ has for dual $K^\oplus = \{(z,\lambda) \mid \|z\|_\star \leq \lambda\}$, where $\|z\|_\star := \sup_{x:\|x\|\leq 1} z^\top x$.*

Exercise 3. *We consider the set of $n \times n$ symmetric real matrices $S_n(\mathbb{R})$.*

1. *Show that $\langle A, B\rangle = \mathrm{tr}(AB)$ is a scalar product on S_n.*
2. *Show that the set of semi-definite positive matrices $K = S_n^+(\mathbb{R})$ is a cone.*
3. *Show that $K = S_n^+(\mathbb{R})$ is self dual (i.e. $K = K^\oplus$ for this scalar product).*

Convex functions

Exercise 4 (Moving average). *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function.*

1. *Show that, $s \mapsto \int_0^1 f(st)dt$ is convex.*
2. *Show that, $\mathbb{R}_+^* \ni T \mapsto 1/T \int_0^T f(t)dt$ is convex.*

Exercise 5 (Partial infimum). *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \bar{\mathbb{R}}$ be a convex function and $C \subset \mathbb{R}^m$ a convex set. Show that the function*

$$g : x \mapsto \inf_{y \in C} f(x,y)$$

is convex.

Exercise 6 (log determinant). *Let, for any $X \in S_n$, $f(X) = \ln(\det(X))$ for $X \succ 0$, $-\infty$ otherwise. Consider, for $Z \succ 0$, and $V \in S_n$, the function $g : t \mapsto f(Z + tV)$.*

1. *Show that $g(t) = \sum_{i=1}^n \ln(1 + t\lambda_i) + f(Z)$, where the λ_i are the eigenvalues of $Z^{-1/2} V Z^{-1/2}$.*
2. *Show that g is concave. Conclude that f is concave.*

Exercise 7 (Perspective function). *Let $\phi : E \to \mathbb{R} \cup \{+\infty\}$. The perspective of ϕ is defined as $\tilde{\phi} : \mathbb{R}_+^* \times E \to \mathbb{R}$ by*

$$\tilde{\phi}(\eta, y) := \eta\phi(y/\eta).$$

Show that ϕ is convex iff $\tilde{\phi}$ is convex.

Fenchel transform and subdifferential

Exercise 8 (Norm). *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and $\|y\|_\star := \sup_{x:\|x\|\leq 1} y\top x$ be its dual norm. Let $f : x \mapsto \|x\|$. Compute $f^\star$ and $\partial f(0)$.*

Exercise 9 (Log sum exp)**.** *We consider* $f(x) := \ln(\sum_{i=1}^n e^{x_i})$.

1. *Show that* f *is convex. Hint : recall Holder's inequality* $x^\top y \leq \|x\|_p \|y\|_q$ *for* $1/p + 1/q = 1$.
2. *Show that* $f^\star(y) = \sum_{i=1}^n y_i \ln(y_i)$ *if* $y \geq 0$ *and* $\sum_i y_i = 1$, $+\infty$ *otherwise.*