Exercises : Convex analysis

February 14, 2022

Convex sets

Exercise 1 (Perspective function)**.** *Let* $P : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ *be the* perspective function *defined as* $P(x,t) = x/t$, *with* $\mathrm{dom}(P) = \mathbb{R}^n \times \mathbb{R}_+^*$.

1. *Show that* $P([(x,s),(y,t)]) = [P((x,s)), P((y,t))]$.
2. *Show that, if* $C \subset \mathbb{R}^n \times \mathbb{R}_+^*$ *is convex, then* $P(C)$ *is convex.*
3. *Show that, if* $C \subset \mathbb{R}^n$, *then* $P^{-1}(C)$ *is convex.*

Exercise 2 (Dual cones)**.** *Recall that, for any set* $K \subset \mathbb{R}^n$, $K^\star := \{y \in \mathbb{R}^n \mid \forall x \in K,\ \langle y, x\rangle \geq 0\}$. *We say that* K *is self dual if* $K^\star = K$.

1. *Show that* $K = \mathbb{R}_+^n$ *is self dual.*
2. *We consider the set of symmetric matrices* S_n *with the scalar product* $\langle A, B\rangle = \mathrm{tr}(AB)$. *Show that* $K = S_n^+(\mathbb{R})$ *is self dual.*
3. *Show that* $K = \{(x,t) \mid \|x\| \leq t\}$ *has for dual* $K^\star = \{(z,\lambda) \mid \|z\|_\star \leq z\}$, *where* $\|z\|_\star := \sup_{x:\|x\|\leq 1} z^\top x$.

Convex functions

Exercise 3 (Moving average)**.** *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a convex function.*

1. *Show that,* $s \mapsto \int_0^1 f(st)dt$ *is convex.*
2. *Show that,* $T \mapsto 1/T \int_0^T f(t)dt$ *is convex.*

Exercise 4 (Partial infimum)**.** *Let* $f : \mathbb{R}^n \times \mathbb{R}^m \to \bar{\mathbb{R}}$ *be a convex function and* $C \subset \mathbb{R}^m$ *a convex set. Show that the function*

$$g : x \mapsto \inf_{y \in C} f(x,y)$$

is convex.

Exercise 5 (log determinant)**.** *Let, for any* $X \in S_n$, $f(X) = \ln(\det(X))$. *Consider, for* $Z \succ 0$, *and* $V \in S_n$, *the function* $g : t \mapsto f(Z + tV)$.

1. *Show that* $g(t) = \sum_{i=1}^n \ln(1 + t\lambda_i) + f(Z)$, *where the* λ_i *are the eigenvalues of* $Z^{-1/2} V Z^{-1/2}$.
2. *Show that* g *is concave. Conclude that* f *is concave.*

Exercise 6 (Perspective function)**.** *Let* $\phi : E \to \mathbb{R} \cup \{+\infty\}$. *The perspective of* ϕ *is defined as* $\tilde{\phi} : \mathbb{R}_+^* \times E \to \mathbb{R}$ *by*

$$\tilde{\phi}(\eta, y) := \eta \phi(y/\eta).$$

Show that ϕ *is convex iff* $\tilde{\phi}$ *is convex.*

Fenchel transform and subdifferential

Exercise 7 (Norm)**.** *Let* $\|\cdot\|$ *be a norm on* $\mathbb{R}^n$ *and* $\|y\|_\star := \sup_{x:\|x\|\leq 1} y^\top x$ *its dual norm. Let* $f : x \mapsto \|x\|$. *Compute* $f^\star$ *and* $\partial f(0)$.

Exercise 8 (Log sum exp)**.** *We consider* $f(x) := \ln(\sum_{i=1}^n e^{x_i})$.

1. *Show that* f *is convex. Hint : recall Holder's inequality* $x^\top y \leq \|x\|_p \|y\|_q$ *for* $1/p + 1/q = 1$.
2. *Show that* $f^\star(y) = \sum_{i=1}^n y_i \ln(y_i)$ *if* $y \geq 0$ *and* $\sum_i y_i = 1$, $+\infty$ *otherwise.*