Exercises: Linear Algebra Recall (for Optimization)

1. Diagonalization and spectrum

Exercise 1 (Spectrum and invertibility). *Let $A \in \mathbb{R}^{n\times n}$.*

(a) Show that A is invertible if and only if $0 \notin \sigma(A)$.

(b) Show that if A is invertible then every eigenvalue of A^{-1} is of the form $1/\lambda$ where $\lambda \in \sigma(A)$.

Exercise 2 (Distinct eigenvalues ⇒ diagonalizable). *Assume $A \in \mathbb{R}^{n\times n}$ has n* distinct *real eigenvalues. Show that A is diagonalizable over $\mathbb{R}$.*

Exercise 3 (Orthogonal diagonalization is special). *Give an example of a diagonalizable (real) matrix A that is* not *orthogonally diagonalizable (i.e., there is no orthogonal Q with $A = Q\Lambda Q^\top$).*

2. PSD order and spectral theorem

Exercise 4 (Loewner order is a partial order). *On S_n (real symmetric matrices), define $A \preceq B$ iff $B - A \succeq 0$. Show that $\preceq$ is a partial order on S_n (reflexive, antisymmetric, transitive).*

Exercise 5 (PSD/PD and eigenvalues). *Let $A \in S_n$ with spectral decomposition $A = Q\Lambda Q^\top$.*

(a) Prove $A \succeq 0$ if and only if all eigenvalues satisfy $\lambda_i \geq 0$.

(b) Prove $A \succ 0$ if and only if all eigenvalues satisfy $\lambda_i > 0$.

(c) Assume $A \succeq 0$. Define $A^{1/2} = Q\Lambda^{1/2}Q^\top$ with $\Lambda^{1/2} = \mathrm{diag}(\sqrt{\lambda_i})$. Show that $A^{1/2}A^{1/2} = A$.

Exercise 6 (Rayleigh quotient extrema). *Let $A \in S_n$ and define $R_A(x) = \dfrac{x^\top Ax}{x^\top x}$ for $x \neq 0$. Show that*

$$\lambda_{\min}(A) = \min_{\|x\|_2=1} x^\top Ax, \qquad \lambda_{\max}(A) = \max_{\|x\|_2=1} x^\top Ax.$$

Exercise 7 (Eigenvalue bounds from PSD order). *Let $A, B \in S_n$ and assume $A \preceq B$. Prove that $\lambda_{\min}(A) \leq \lambda_{\min}(B)$ and $\lambda_{\max}(A) \leq \lambda_{\max}(B)$.*

3. Orthogonal projectors

Exercise 8 (Projector onto a subspace given an orthonormal basis). *Let $Q \in \mathbb{R}^{n\times k}$ have orthonormal columns ($Q^\top Q = I_k$). Define $P = QQ^\top$.*

(a) Prove that P is an orthogonal projector.

(b) Prove that $\mathrm{Im}(P) = \mathrm{Im}(Q)$.

(c) Show that for all x, Px is the unique minimizer of $\min_{y\in \mathrm{Im}(Q)} \|x - y\|_2$.

Exercise 9 (Projection onto a hyperplane). *Let $a \in \mathbb{R}^n$ with $a \neq 0$ and consider the hyperplane $H = \{x \in \mathbb{R}^n : a^\top x = b\}$.*

(a) Show that the Euclidean projection of x_0 onto H is

$$\Pi_H(x_0) = x_0 - \frac{a^\top x_0 - b}{\|a\|_2^2}\, a.$$

(b) Deduce the projector onto the subspace $\{x : a^\top x = 0\}$.

4. Norms and inner products

Exercise 10 (Norm inequalities). *Let $x \in \mathbb{R}^n$. Prove*

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\,\|x\|_2 \leq n\,\|x\|_\infty.$$

Exercise 11 (Cauchy–Schwarz equality case). *Let $\langle\cdot,\cdot\rangle$ be an inner product and $\|x\| = \sqrt{\langle x, x\rangle}$. Prove that for $x \neq 0$ and $y \neq 0$, equality in $|\langle x, y\rangle| \leq \|x\|\,\|y\|$ holds if and only if $y = \alpha x$ for some $\alpha \in \mathbb{R}$.*

Exercise 12 ($\|\cdot\|_Q$ norm and eigenvalue bounds)**.** *Let $Q \in S_n^+$ and define $\|x\|_Q := \sqrt{x^\top Qx}$.*

(a) Show that $\|\cdot\|_Q$ is a norm if and only if $Q \succ 0$.

(b) Assume $Q \succ 0$. Show that

$$\|x\|_Q \le \sqrt{\lambda_{\max}(Q)}\,\|x\|_2, \qquad \|x\|_2 \le \frac{1}{\sqrt{\lambda_{\min}(Q)}}\|x\|_Q.$$

5. Dual norms and operator norms

Exercise 13 (Dual norm basics)**.** *Let $\|\cdot\|$ be a norm and define its dual norm by $\|y\|_\star := \sup_{\|x\|\le 1} y^\top x$.*

(a) Prove the generalized Cauchy–Schwarz inequality: $|y^\top x| \le \|y\|_\star\,\|x\|$.

(b) Compute the duals of $\|\cdot\|_2$, $\|\cdot\|_1$, and $\|\cdot\|_\infty$.

Exercise 14 (Induced operator norm)**.** *Given a vector norm $\|\cdot\|$, define*

$$\|A\|_{op} := \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|}.$$

(a) Show that $\|A\|_{op} = \sup_{\|x\|\le 1}\|Ax\|$.

(b) Prove submultiplicativity: $\|AB\|_{op} \le \|A\|_{op}\,\|B\|_{op}$.

(c) For the Euclidean norm, show that $\|A\|_2^2 = \lambda_{\max}(A^\top A)$.

6. Linear systems and factorizations

Exercise 15 (LU and triangular solves)**.** *Assume $A = LU$ with L lower triangular and U upper triangular, both invertible. Show that solving $Ax = b$ reduces to two triangular solves.*

Exercise 16 (Cholesky and SPD)**.** *(a) Let $A \in S_n$ and assume $A = LL^\top$ for some invertible lower triangular L. Prove that $A \succ 0$.*

(b) Compute the Cholesky factorization of $A = \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}$.

(c) Reuse your factorization to solve $Ax = b$ for $b = (2,1)^\top$ and for $b = (0,1)^\top$.

7. Least squares problems

Exercise 17 (Solve least squares via thin QR)**.** *Let $A \in \mathbb{R}^{m\times n}$ with $m \ge n$ and assume A has full column rank. Let $A = QR$ be a thin QR factorization with $Q \in \mathbb{R}^{m\times n}$, $Q^\top Q = I_n$, and $R \in \mathbb{R}^{n\times n}$ upper triangular (hence invertible).*

(a) Let $P := QQ^\top$. Prove the identity

$$\|Ax-b\|_2^2 = \|Rx-Q^\top b\|_2^2 + \|(I-P)b\|_2^2 \qquad \forall x \in \mathbb{R}^n.$$

(b) Deduce that the unique minimizer of $\min_x \|Ax-b\|_2^2$ satisfies

$$Rx^\star = Q^\top b,$$

and explain why this can be solved by backward substitution.

Exercise 18 (Normal equations square the conditioning)**.** *Assume $A \in \mathbb{R}^{m\times n}$ has full column rank. Recall $\kappa_2(A) = \|A\|_2\|A^\dagger\|_2$ (where $A^\dagger = (A^\top A)^{-1}A^\top$) and $\kappa_2(A^\top A) = \|A^\top A\|_2\|(A^\top A)^{-1}\|_2$.*

(a) Show that $\|A^\top A\|_2 = \|A\|_2^2$.

(b) Show that $\|(A^\top A)^{-1}\|_2 = \|A^\dagger\|_2^2$ and deduce

$$\kappa_2(A^\top A) = \kappa_2(A)^2.$$

(c) Interpret why this suggests avoiding normal equations in finite precision.

8. SVD, conditioning, and numerical stability

Exercise 19 (SVD on a simple matrix)**.** *Let $A = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}$ with $\varepsilon > 0$.*

(a) Compute the singular values of A.

(b) Compute $\kappa_2(A)$.