Convex Optimization Exam

03/06/2022
3 hours – documents allowed
Answers in English or French

The exam is made of 4 independent exercises, in roughly increasing difficulty. If necessary, you can admit the results of previous questions. When using the recalls, cite them. "Classifying" an optimization problem consists in precising in which of the category presented in chapter 5 it falls (LP, QP, QCQP, SOCP, SDP, unconstrained or not, differentiable or not, continuous or not, convex or not).

Some useful recalls

i) Recall that a step τ is deemed admissible in the backtracking step rules if $f(x_k+\tau d_k) \leq f(x_k)+\alpha\tau g_k^\top d_k$ for some $\alpha \in]0, 1/2[$.

ii) For $p, q \in]1, +\infty[$, $1/p + 1/q = 1$, we also have $q = \frac{p}{p-1}$ and $\frac{q}{p} + 1 = q$

iii) Let S_n be the set of symmetric real valued matrices. Then all $A \in S_n$ is diagonalizable. We denote S_n^+ (resp. S_n^{++}) the set of semidefinite (resp. definite) symmetric matrix, *i.e.,* all eigenvalues are non-negative (resp. strictly positive). For $A, B \in S_n$, $A \preceq B$ iff $B - A$ is semidefinite positive (denoted $B - A \succeq 0$.

iv) S_n is an euclidean space, whose canonical scalar product is $\langle A, B\rangle = tr(AB)$.

Exercice 1: Warm-up *2 points*

(a) (1 point) In Figure 1 we represent level set of some function. Are there some cases where the function cannot be convex ? Briefly justify.

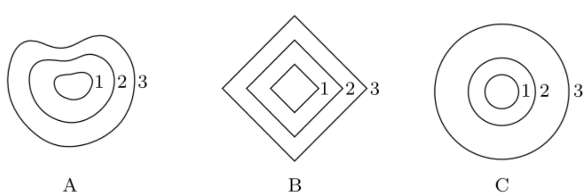

Figure 1: level set of potentially convex function ?

(b) (1 point) Consider $f : \mathbb{R}^2 \to \mathbb{R}$, $f : x \mapsto x_1^2 + x_2^2$, and $C = \{x_1 + x_2 \geq 1\}$. Gives, for every $x \in \mathbb{R}_2$ the normal cone $N_C(x)$. Use this to solve $\min_{x\in C} f(x)$ through the convex optimality condition.

Exercice 2: Projection over the L_1 ball *5 points*

Let $a \in \mathbb{R}^n$. We consider the following optimization problem.

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^n} & \frac{1}{2}\|x-a\|^2 \\ \text{s.t.} & \|x\|_1 \leq 1 \end{array}$$

(a) (1 point) Classify this problem, and justify that we have strong duality. Justify existence and unicity of optimal solution.

(b) (1 point) Write the (Lagrangian) dual problem as

$$\max_{\lambda\geq 0} g(\lambda) := \sum_{k=1}^{n} g_k(\lambda) - \lambda$$

where g_k should be given as analytical formula (*i.e.*, without "min").

(c) (1 point) Show that $g'(\lambda) = \sum_{k=1}^{n}(|a_k| - \lambda)^+ - 1$.

(d) (1 point) Suggest an efficient method to find the optimal multiplier $\lambda^\sharp$.

(e) (1 point) Explain how to obtain the optimal primal solution $x^\sharp$ from $\lambda^\sharp$.

Exercice 3: Unit step in Quasi Newton's Method *3 points*

We consider a C^2 strongly-convex function $f : \mathbb{R}^n \to \mathbb{R}$, and the following algorithm, for given x_0, $x_{k+1} = x_k + t_k d_k$ where, for all $k \in \mathbb{N}, d_k = -M_k^{-1} g_k$, $g_k := \nabla f(x_k)$ and M_k is a symmetric definite positive matrix such that

$$d^\top M_k d \geq d^\top \nabla^2 f(x_k) d + o(\|d\|^2)$$

(a) (1 point) Show that this algorithm is a descent algorithm.

(b) (2 points) Assume that x_k converges toward the minimizer of f. Show that there exists K such that, for all $k \geq K$, $t_k = 1$ is admissible for backtracking step rule.

Exercice 4: Minimizing linear functions on a ball *4 points*

For $c \in \mathbb{R}^n$ we are interested in finding the solution of

$$\min_{x\in\mathbb{R}^n} \left\{ c^\top x \quad | \quad \|x\|_p \leq 1 \right\}$$

to prove Hölder inequality, *i.e.*,, $|x^\top y| \leq \|x\|_p \|y\|_q$ for $p, q \in]1, +\infty[$ such that $1/p + 1/q = 1$.

(a) (1 point) Reformulate the problem as a differentiable problem, and write the KKT conditions.

(b) (1 point) Find the optimal solution for $1 < p < +\infty$.

(c) (1 point) Show that $\displaystyle\min_{\|x\|_p\leq 1} c^\top x = -\|x\|_q$.

(d) (1 point) Deduce Hölder inequality from the previous question.

Exercice 5: Sum of largest eigenvalues *6 points*

We consider the function $f : S_n \to \mathbb{R}$ given as the sum of the $r \leq n$ largest eigenvalues, that is

$$f(A) = \sum_{k=1}^{r} \lambda_r(A)$$

where $\lambda_1 \geq \lambda_2 \geq \dots \lambda_n$ are the eigenvalues of A.

(a) (2 points) Show that

$$\begin{aligned} f(A) = \max_{X\in S_n} \quad & \langle A, X \rangle \\ \text{s.t.} \quad & tr(X) = r \\ & 0 \preceq X \preceq I \end{aligned}$$

Classify this problem.

(b) (1 point) Show that f is convex

(c) (3 points) Consider $A(x) := \sum_{i=1}^{K} x_i A_i$ where $A_i \in S_n$, and the problem

$$\min_{x\in\mathbb{R}^K} \quad f(A(x))$$

Using duality, reformulate this problem as an SDP.