1 Warm up

(a) (2 points) We consider

(b) (2 points) We consider a differentiable convex optimization problem

$$(\mathcal{P}) \quad \min_{x\in\mathbb{R}^n} \quad f(x) \qquad s.t. \quad g_i(x) \le 0 \qquad \forall i \in [m]$$

and a family of problem, indexed by $r > 0$,

$$(\mathcal{P}_r) \quad \min_{x\in\mathbb{R}^n} f(x) + \sum_{i=1}^{m} r e^{g_i(x)/r}.$$

What relation can you see between $(\mathcal{P}_r)$ and $\mathcal{P}$? What is the advantage of $(\mathcal{P}_r)$? The trade-off over the choice of r?

(c) (1 point) Write a SGD step for minimizing $f(\theta) = \frac{1}{n}\sum_{i=1}^n f_i(\theta)$. If each f_i is strongly convex, can we ensure convergence toward the global minimum of f?

2 Fast dynamic programming

We are interested in finding the optimal policy to manage a battery. Every 15mn, the operator can sell (by discharging energy from the battery) or buy electricity (by charging energy to the battery) to an external network. The electricity price varies along time, and is here modeled with a probabilistic model. The battery has given capacity and maximal rate of charge/discharge.

We introduce the following parameters:

- T: time horizon.
- $C > 0$: battery capacity [MWh].
- $P > 0$: maximum battery charge/discharge in one hour [MWh].
- $0 < \eta_p < 1$: charge efficiency.
- $0 < \eta_d < 1$: discharge efficiency.
- λ_t: electricity price at time t [€/MWh].
- $c \ge 0$: tax paid when injecting energy onto the network [€/MWh].

Our decision variables are:

- x_t: State-of-Charge (SoC) of the battery at time t [MWh].
- p_t: Energy discharged from the battery between t and $t+1$ [MWh].
- b_t: Energy charged to the battery between t and $t+1$ [MWh].

The operational constraints write, for $t = 1, \cdots, T$,

$$0 \leq x_t \leq C, \quad 0 \leq p_t \leq P, \quad 0 \leq b_t \leq P. \tag{1}$$

The (discrete) dynamics of the battery writes

$$x_{t+1} = x_t - \frac{1}{\eta_p} p_t + \eta_d b_t \ . \tag{2}$$

We suppose that the battery is initially discharged: $x_0 = 0$. We aim at minimizing the operating cost between time $t = 1$ and time $t = T$:

$$\sum_{t=1}^{T} \Big(\lambda_t (b_t - p_t) + c p_t \Big) \ . \tag{3}$$

We recall the following theorem.

Theorem 1 (Sensitivity analysis)**.** *Let the value function $V(\cdot)$ defined as $V(b) = \min_x \ c^\top x$ s.t. $Ax \leq b$ and $(x(b), \lambda(b))$ a primal-dual solution of the linear program. For all b such that $V(b) < +\infty$, we have $-\lambda(b) \in \partial V(b)$.*

2.1 Deterministic problem

First, we suppose the electricity prices $\lambda = (\lambda_1, \ldots, \lambda_T)$ are known in advance.

1. (1 point) Write an optimisation problem depending on $\lambda = (\lambda_1, \ldots, \lambda_T)$ whose solution is the optimal management policy $(x_t, b_t, p_t)_{t=1,\cdots,T}$. We note the optimal objective $v(\lambda)$.

Solution:

$$\begin{aligned} v(\lambda) = \min \ & \sum_{t=1}^{T} \Big(\lambda_t (b_t - p_t) + c p_t \Big) \\ \text{s.t.} \ & x_{t+1} = x_t - \frac{1}{\eta_p} p_t + \eta_d b_t \\ & 0 \leq x_t \leq C \\ & 0 \leq b_t \leq P \\ & 0 \leq p_t \leq P \\ & x_0 = 0 \end{aligned} \tag{4}$$

2. (1 point) Classify this problem and name two efficient optimization algorithms to solve it.

Solution: The LP can be solved using the simplex algorithm or the interior-point method.

2.2 Stochastic problem

In what follows, we will note all the random variables in **bold**.

We assume that the energy prices $\boldsymbol{\lambda}_t$ follow for all $t \in [T]$,

$$\log(\boldsymbol{\lambda}_t) = \log(\overline{\lambda}_t) + \boldsymbol{\xi}_t \ ; \tag{5}$$

where $\overline{\lambda}_t$ for all $t \in [T]$ are given parameters, and $\boldsymbol{\xi} = \{\boldsymbol{\xi}_t\}_{t \in [T]}$ is a stationary Markov chain. At each time step t the random variable $\boldsymbol{\xi}_t$ can take N distinct values $\{\xi_1, \ldots, \xi_N\}$. The transition between $\boldsymbol{\xi}_t$ and $\boldsymbol{\xi}_{t+1}$ is encoded by the conditional probabilities $\{p_{ij}\}_{i,j=1,\ldots,N}$:

$$\mathbb{P}[\boldsymbol{\xi}_{t+1} = \xi_j \mid \boldsymbol{\xi}_t = \xi_i] = p_{ij} \ . \tag{6}$$

4. (1 point) Write the stochastic optimization problem aiming at minimizing the expected cost.

Solution:

$$
\begin{aligned}
\min \quad & \mathbb{E}\Big[\sum_{t=1}^{T} \Big(\boldsymbol{\lambda}_t(\boldsymbol{b}_t - \boldsymbol{p}_t) + c\boldsymbol{p}_t\Big)\Big] & \\
\text{s.t.} \quad & \boldsymbol{x}_{t+1} = \boldsymbol{x}_t - \frac{1}{\eta_p}\boldsymbol{p}_t + \eta_d \boldsymbol{b}_t & \mathbb{P} - a.s. \\
& 0 \leq \boldsymbol{x}_t \leq C & \forall t, \mathbb{P} - a.s. \\
& 0 \leq \boldsymbol{b}_t \leq P & \forall t, \mathbb{P} - a.s. \\
& 0 \leq \boldsymbol{p}_t \leq P & \forall t, \mathbb{P} - a.s. \\
& \sigma(\boldsymbol{p}_t) \subset \mathcal{F}_{t+1}, \quad \sigma(\boldsymbol{b}_t) \subset \mathcal{F}_{t+1} & \forall t, \\
& x_0 = 0 &
\end{aligned}
\tag{7}
$$

We want to solve the stochastic program using the Stochastic Dynamic Programming algorithm, adapted here to the Markovian setting. The final value function is defined as $V_{T+1}(x) = 0$. For $t \in [T]$ and $j \in [N]$, we define $Q_{t,j}(\cdot)$ as

$$
\begin{aligned}
Q_{t,j}(x) = \min_{x^+, p, b} \; & \lambda_{t,j}(b - p) + cp + V_{t+1,j}(x^+) \\
\text{s.t.} \quad & x^+ = x - \frac{1}{\eta_p} p + \eta_d b \\
& 0 \leq b \leq \overline{b}(x) \\
& 0 \leq p \leq \overline{p}(x)
\end{aligned}
\tag{8}
$$

with $\overline{b}(x) := \min\Big(\frac{c-x}{\eta_d}, P\Big)$ and $\overline{p}(x) := \min\Big(\eta_p x, P\Big)$. Using (8), the value functions $V_{t,i}$ satisfies the Bellman's recursive equations

$$
V_{t,i}(x) = \sum_{j=1}^{N} p_{ij} Q_{t,j}(x) \; , \quad \forall x \in [0, C] \; . \tag{9}
$$

For a given x, we note by $g_{t,i}(x)$ an element of the subdifferential of the convex function $V_{t,i}$: $g_{t,i}(x) \in \partial V_{t,i}(x)$. Such an element is called a *costate*.

5. (2 points) We introduce the Lagrangian for the optimization problem (8):

$$
L(x^+, b, p; \beta, \gamma, \pi) = \lambda_{t,j}(b-p) + cp + V_{t+1,j}(x^+) - \underline{\beta} b + \overline{\beta}(b - \overline{b}(x)) - \underline{\gamma} p + \overline{\gamma}(p - \overline{p}(x)) + \pi(x - \frac{1}{\eta_p} p + \eta_d b - x^+) \; .
$$

Write the KKT conditions[1] of (8) and show they depend on the future costate $g_{t+1,j}(x^+) \in \partial V_{t+1,j}(x^+)$.

Solution: Using the Lagrangian, the KKT conditions of the LP problem writes out, for every

[1]You can act as if $V_{t+1,j}$ was differentiable and $g_{t+1,j}(x^+) = V'_{t+1,j}(x^+)$.

$g_{t+1,j}(x^+) \in \partial V_{t+1,j}(x^+)$:

$$\begin{aligned}
& g_{t+1,j}(x^+) - \pi = 0 && [\partial L/\partial x^+ = 0] \\
& \lambda_{t,j} - \underline{\beta} + \overline{\beta} + \eta_d \pi = 0 && [\partial L/\partial b = 0] \\
& -\lambda_{t,j} + c - \underline{\gamma} + \overline{\gamma} - \frac{1}{\eta_p}\pi = 0 && [\partial L/\partial p = 0] \\
& x^+ - x + \frac{1}{\eta_p} p - \eta_d b = 0 && \text{primal admissibility} \\
& 0 \le b \le \overline{b}(x) \\
& 0 \le p \le \overline{p}(x) \\
& 0 \le b \perp \underline{\beta} \ge 0 \\
& 0 \le (\overline{b}(x) - b) \perp \overline{\beta} \ge 0 \\
& 0 \le p \perp \underline{\gamma} \ge 0 \\
& 0 \le (\overline{p}(x) - p) \perp \overline{\gamma} \ge 0
\end{aligned} \tag{10}$$

6. (1 point) Suppose that for all t, the price is non-negative: $\lambda_t \ge 0$. Is it a reasonable assumption? Under this assumption, prove that we cannot charge and discharge the battery simultaneously.

Solution: We suppose that both $p > 0$ and $b > 0$. According to the KKT conditions explicited in the previous question, the complementarity conditions imply $\underline{\beta} = 0$, $\overline{\beta} > 0$ and $\underline{\gamma} = 0$, $\overline{\gamma} > 0$. We deduce:

$$\pi = \frac{1}{\eta_d}\Big(-\lambda_{t,j} - \overline{\beta}\Big) = \eta_p \Big(-\lambda_{t,j} + c + \overline{\gamma}\Big) \tag{11}$$

or equivalently,

$$(1 - \eta_p \eta_d)\lambda_{t,j} = -\eta_p \eta_d c - \eta_p \eta_d \overline{\gamma} - \overline{\beta} \tag{12}$$

The left-hand-side is positive (as $\eta_p \eta_d < 1$) and the right-hand-side is negative, leading to a contradiction. Hence we cannot have both $p > 0$ and $b > 0$.

7. (2 points) Deduce from the previous question that only three situations can occur: (i) $b > 0, p = 0$, (ii) $b = 0, p = 0$ and (iii) $b = 0, p > 0$. Use the KKT conditions to characterize the three alternatives using the problem's data and the multiplier π. Specify on each case if the battery is charging, discharging or idle.

Solution: According to the previous question, we cannot have both $p > 0$ and $b > 0$. As $(b, p) \ge 0$, we obtain the three alternatives listed above. From the KKT conditions, we get:

$$\overline{\beta} - \underline{\beta} = -\eta_d \pi - \lambda_{t,j} \ , \quad \overline{\gamma} - \underline{\gamma} = \frac{1}{\eta_p}\pi - c + \lambda_{t,j} \tag{13}$$

As we have both $(\underline{\gamma}, \overline{\gamma}) \ge 0$ and $(\underline{\beta}, \overline{\beta}) \ge 0$, we deduce that

$$\begin{cases}
\overline{\beta} = \max\{0, -\eta_d \pi - \lambda_{t,j}\} \ , \\
\underline{\beta} = \max\{0, \eta_d \pi + \lambda_{t,j}\} \ , \\
\overline{\gamma} = \max\{0, \frac{1}{\eta_p}\pi - c + \lambda_{t,j}\} \ , \\
\underline{\gamma} = \max\{0, -\frac{1}{\eta_p}\pi + c - \lambda_{t,j}\} \ .
\end{cases} \tag{14}$$

We observe that the multipliers (14) are piecewise affine. Depending on the value of $\pi = v_{t+1,j}(x^+)$, we get the following situations:

- if $\pi \geq \eta_p(c - \lambda_{t,j})$, we have $(\overline{\gamma}, \underline{\beta}) \geq 0$ and $(\underline{\gamma}, \overline{\beta}) = 0$ implying $p \geq 0$ and $b = 0$: the battery is **discharging**.
- if $-\frac{1}{\eta_d}\lambda_{t,j} < \pi < \eta_p(c - \lambda_{t,j})$, we have $(\underline{\gamma}, \underline{\beta}) > 0$, implying $p = 0$ and $b = 0$: the SoC remains **constant**.
- if $\pi \leq -\frac{1}{\eta_d}\lambda_{t,j}$, we have $(\underline{\gamma}, \overline{\beta}) \geq 0$ and $(\overline{\gamma}, \underline{\beta}) = 0$ implying $p = 0$ and $b \geq 0$: the battery is **charging**.

8. (1 point) Let $\pi_+ = g_{t+1,j}\big(x + \eta_d \overline{b}(x)\big)$, $\pi_0 = g_{t+1,j}(x)$ and $\pi_- = g_{t+1,j}\Big(x - \frac{1}{\eta_p}\overline{p}(x)\Big)$. Show that $\pi_- \leq \pi_0 \leq \pi_+$.

Solution: The value function $V_{t+1,j}$ is convex, hence $g_{t+1,j}(\cdot)$ is non-decreasing and:

$$g_{t+1,j}\left(x - \frac{1}{\eta_p}\overline{p}(x)\right) \leq g_{t+1,j}(x) \leq g_{t+1,j}(x + \eta_d \overline{b}(x)) \ . \tag{15}$$

9. (1 point) Using the two previous questions, deduce the optimal policy solution of (8) as function of the SoC x and the electricity price $\lambda_{t,j}$. For a fixed $\lambda_{t,j}$, is the optimal policy piecewise constant? Piecewise linear?

Solution: According to the KKT conditions, we have $\pi = g_{t+1,j}(x^+)$. The next SoC x^+ takes values in the interval $[x - \frac{1}{\eta_p}\overline{p}(x), x + \eta_d \overline{b}(x)]$. Using the previous question, we deduce that only one of the five conditions can occur:

1. If $\pi_+ \leq -\frac{1}{\eta_d}\lambda_{t,j}$, then $(b, p) = (\overline{b}(x), 0)$.
2. If $\pi_0 \leq -\frac{1}{\eta_d}\lambda_{t,j} \leq \pi_+$, then $(b, p) = \big((\pi^{-1}(-1/\eta_d \lambda_{t,j}) - x)/\eta_d, 0\big)$.
3. If $-\frac{1}{\eta_d}\lambda_{t,j} \leq \pi_0 \leq \eta_p(c - \lambda_{t,j})$ then $(b, p) = (0, 0)$.
4. If $\pi_- \leq \eta_p(c - \lambda_{t,j}) \leq \pi_0$ then $(b, p) = (0, (\pi^{-1}(\eta_p(c - \lambda_{t,j}) - x) \times \eta_p)$.
5. If $\eta_p(c - \lambda_{t,j}) \leq \pi_-$ then $(b, p) = (0, \overline{p}(x))$.

The optimal policy is piecewise constant.

10. (1 point) Let $q_{t,j}(x) \in \partial Q_{t,j}(x)$. Apply Theorem 1 to (8) to show that $q_{t,j}(x)$ depends on the dual multipliers (π, β, γ).

Solution: Using Theorem 1, we have that

$$q_{t,j}(x) = \pi + \frac{1}{\eta_d}\overline{\beta} - \eta_p \overline{\gamma} \ . \tag{16}$$

11. (2 points) Deduce from the previous question that the costates $\{g_{t,i}\}_{t,i}$ satisfy a set of recursive equations, analogous to the Dynamic Programming equations (9).

Solution: As $V_{t,i} = \sum_{j=1}^{N} p_{ij} Q_{t,j}$, we deduce that $v_{t,i}$ satisfies the recursive equation:

$$g_{t,i}(x) = \sum_{j=1}^{N} p_{ij} q_{t,j}(x) \ . \tag{17}$$

Using the previous question, we can compute the sensitivity $q_{t,j}(x)$ analytically:

$$q_{t,j}(x) = \pi_j + \frac{1}{\eta_d} \overline{\beta}_j - \eta_p \overline{\gamma}_j \ , \tag{18}$$

with $(\pi_j, \beta_j, \gamma_j)$ the dual multipliers associated to (8). The KKT conditions give that the multiplier π_j depends on the future state x_{j+}:

$$\pi_j = v_{t+1,j}(x_{j+}) \ . \tag{19}$$