Convex optimization - Take home exam 2024
—To be returned by May, 17th—

Let $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$ given data. We define the primal linear program (LP) and its dual as:

$$\min_{x\in\mathbb{R}^n} c^\top x \quad \text{s.t.} \quad Ax = b\ ,\ x \geq 0\ , \qquad \max_{\lambda\in\mathbb{R}^m, s\in\mathbb{R}^n} b^\top \lambda \quad \text{s.t.} \quad A^\top \lambda + s = c\ ,\ s \geq 0\ . \tag{1}$$

We recall the following theorem.

Theorem 1 (Goldman-Tucker)**.** *Suppose the primal and dual LP* (1) *are feasible. There exists at least one primal-dual solution* $(x^\star, \lambda^\star, s^\star)$ *satisfying strict complementarity:* $x^\star + s^\star > 0$.

1 Self duality

Let $f \in \mathbb{R}^n$, $g \in \mathbb{R}^m$ two vectors and $C \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n\times m}$ and $E \in \mathbb{R}^{m\times m}$ three matrices. We define the linear program (LP)

$$\begin{aligned} \min_{u\in\mathbb{R}^n, v\in\mathbb{R}^m}\ & f^\top u + g^\top v \\ \text{subject to}\quad & Cu + Dv \geq -f && [\rightsquigarrow \widetilde{u}] \\ & -D^\top u + Ev = -g && [\rightsquigarrow \widetilde{v}] \\ & u \geq 0 && [\rightsquigarrow w] \end{aligned} \tag{2}$$

We suppose C and D are skew-symmetric matrices: $C^\top = -C$ and $E^\top = -E$. We will note by $(\widetilde{u}, \widetilde{v}, w)$ the Lagrangian multipliers attached respectively to the constraints $Cu + Dv \geq -f$, $-D^\top u + Ev = -g$ and $u \geq 0$.

1. Write the KKT conditions of the LP (2).
2. Show that the LP (2) is self-dual, in the sense that its Lagrangian dual problem is exactly (2).
3. Show that the LP (2) is equivalent to the linear complementarity problem (LCP):

$$\text{Find} \quad (u, v, w) \quad \text{such that} \quad \begin{cases} \begin{bmatrix} w \\ 0 \end{bmatrix} = \begin{bmatrix} C & D \\ -D^\top & E \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} f \\ g \end{bmatrix}\ , \\ (u, w) \geq 0\ ,\ u^\top w = 0\ . \end{cases} \tag{4}$$

4. Use the Goldman-Tucker theorem to prove that if the LP (2) is feasible, then the LCP (4) has a strictly complementary solution: i.e., there exists (u, v, w) solution of (4) such that $u + w > 0$.

2 Simplified homogeneous self-dual (HSD) embedding

We introduce the following LP:

$$\begin{aligned} \min_{x,\lambda,\tau,s,\kappa}\ & 0 \\ \text{s.t.}\ & \begin{bmatrix} 0 & -A^\top & c \\ A & 0 & -b \\ -c^\top & b^\top & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \\ \tau \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ \kappa \end{bmatrix} \\ & (x, \tau, s, \kappa) \geq 0\ . \end{aligned} \tag{5}$$

1. Show that the LP (5) has a trivial solution, and all feasible points are optimal and satisfy

$$s^\top x + \kappa\tau = 0\ . \tag{6}$$

2. Justify that if $(x, \lambda, \tau, s, \kappa)$ is solution of the LP (5), then for all $t > 0$ the vector $(tx, t\lambda, t\tau, ts, t\kappa)$ is also a solution of (5).

3. Suppose that $(x, \lambda, \tau, s, \kappa)$ is solution of (5), with $\tau > 0$ and $\kappa = 0$. Show that $(\hat{x}, \hat{\lambda}, \hat{s}) := (x/\tau, \lambda/\tau, s/\tau)$ is a solution of the original LP (1).

4. Show that the LP (5) has the same structure as the self-dual LP (2). Explicit the matrix C, D, E and the vectors f, g. Deduce that the original LP (2) is equivalent to the LCP:

$$\text{Find} \quad (x, \lambda, \tau, s, \kappa) \quad \text{such that} \quad \begin{cases} \begin{bmatrix} 0 & -A^\top & c \\ A & 0 & -b \\ -c^\top & b^\top & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \\ \tau \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ \kappa \end{bmatrix} \\ (x, \tau, s, \kappa) \geq 0 \; , \quad s^\top x + \kappa\tau = 0 \; . \end{cases} \tag{7}$$

3 Generalization to conic programming

We recall the following result.

Theorem 2 (Moreau decomposition)**.** *Let K be a closed convex cone and $K^\oplus$ its positive dual cone. We note P_K (resp. $P_{-K^\oplus}$) the Euclidean projection onto K (resp. $-K^\oplus$). For $x, y, z \in \mathbb{R}^n$, the following statements are equivalent:*

1. $z = x + y$ for $x \in K$, $y \in -K^\oplus$ and $x^\top y = 0$.

2. $x = P_K(z)$ and $y = P_{-K^\oplus}(z)$.

1. Let $K = \mathbb{R}^n_+ \times \mathbb{R}^m \times \mathbb{R}_+$.
   (a) Give the expression of the dual cone $K^\oplus$.
   (b) For a matrix Q to be explicited, deduce that we can rewrite (7) more compactly as

$$\text{Find} \quad (u, v) \quad \text{such that} \quad \begin{cases} v = Qu \; , \\ (u, v) \in K \times K^\oplus \; . \end{cases} \tag{8}$$

   (c) Use the Moreau decomposition theorem to prove that (8) is equivalent to finding $z \in \mathbb{R}^{n+m+1}$ such that

$$-P_{-K^\oplus}(z) = QP_K(z) \; . \tag{9}$$

2. **[Bonus question:]** Let $\mathcal{C} \subset \mathbb{R}^n$ a proper cone. How to adapt the homogeneous self-dual embedding to solve the following conic problem?

$$\min_{x \in \mathbb{R}^n} c^\top x \quad \text{s.t.} \quad Ax = b \; , \; x \in \mathcal{C} \; . \tag{10}$$