Convex optimization - Take home exam 2024
—To be returned by May, 17th—

Let $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$ given data. We define the primal linear program (LP) and its dual as:

$$\min_{x\in\mathbb{R}^n} c^\top x \quad \text{s.t.} \quad Ax = b\ ,\ x \geq 0\ , \qquad \max_{\lambda\in\mathbb{R}^m, s\in\mathbb{R}^n} b^\top \lambda \quad \text{s.t.} \quad A^\top\lambda + s = c\ ,\ s \geq 0\ . \tag{1}$$

We recall the following theorem.

Theorem 1 (Goldman-Tucker)**.** *Suppose the primal and dual LP (1) are feasible. There exists at least one primal-dual solution $(x^\star, \lambda^\star, s^\star)$ satisfying strict complementarity: $x^\star + s^\star > 0$.*

1 Self duality

Let $f \in \mathbb{R}^n$, $g \in \mathbb{R}^m$ two vectors and $C \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n\times m}$ and $E \in \mathbb{R}^{m\times m}$ three matrices. We define the linear program (LP)

$$\begin{aligned} \min_{u\in\mathbb{R}^n, v\in\mathbb{R}^m} \quad & f^\top u + g^\top v \\ \text{subject to} \quad & Cu + Dv \geq -f & [\rightsquigarrow \widetilde{u}] \\ & -D^\top u + Ev = -g & [\rightsquigarrow \widetilde{v}] \\ & u \geq 0 & [\rightsquigarrow w] \end{aligned} \tag{2}$$

We suppose C and E are skew-symmetric matrices: $C^\top = -C$ and $E^\top = -E$. We will note by $(\widetilde{u}, \widetilde{v}, w)$ the Lagrangian multipliers attached respectively to the constraints $Cu + Dv \geq -f$, $-D^\top u + Ev = -g$ and $u \geq 0$.

1. Write the KKT conditions of the LP (2).

Solution: We note $\widetilde{u}$, $\widetilde{v}$ and w the Lagrangian multipliers associated respectively to $Cu+Dv \geq -f$, $-D^\top u + Ev = -g$ and $u \geq 0$. The Lagrangian of (2) is

$$L(u, v, \widetilde{u}, \widetilde{v}, w) = f^\top u + g^\top v - \widetilde{u}^\top(f + Cu + Dv) - \widetilde{v}^\top(g - D^\top u + Ev) - w^\top u\ .$$

The KKT conditions are:

$$\begin{cases} f - C^\top \widetilde{u} + D\widetilde{v} - w = 0 & (\nabla_u L(\cdot) = 0) \\ g - D^\top \widetilde{u} - E^\top \widetilde{v} = 0 & (\nabla_v L(\cdot) = 0) \\ f + Cu + Dv \geq 0 & (\nabla_{\widetilde{u}} L(\cdot) = 0) \\ g - D^\top u + Ev = 0 & (\nabla_{\widetilde{v}} L(\cdot) = 0) \\ \widetilde{u}^\top(Cu + Dv + f) = 0 & \text{(complementarity)} \\ (u, \widetilde{u}, w) \geq 0 & \text{(dual admissibility)} \end{cases}$$

2. Show that the LP (2) is self-dual, in the sense that its Lagrangian dual problem is exactly (2).

Solution: The dual problem is defined as

$$\max_{\widetilde{u},\widetilde{v},w} \left(\min_{u,v} L(u, v, \widetilde{u}, \widetilde{v}, w) \right) \quad \text{subject to} \quad (\widetilde{u}, w) \geq 0\ .$$

After some reordering, we obtain:

$$L(u,v,\widetilde{u},\widetilde{v},w) = -f^\top \widetilde{u} - g^\top \widetilde{v} + (f - C^\top \widetilde{u} + D\widetilde{v} - w)^\top u + (g - D^\top \widetilde{u} - E^\top \widetilde{v})^\top v \ .$$

We have $\min_{u,v} L(u,v,\widetilde{u},\widetilde{v},w) > -\infty$ only if $f - C^\top \widetilde{u} + D\widetilde{v} - w = 0$ and $g - D^\top \widetilde{u} - E^\top \widetilde{v} = 0$. Hence the dual problem writes

$$\begin{aligned} \max_{\widetilde{u},\widetilde{v},w} \ & - f^\top \widetilde{u} - g^\top \widetilde{v} \\ \text{subject to} \ & f - C^\top \widetilde{u} + D\widetilde{v} - w = 0 \\ & g - D^\top \widetilde{u} - E^\top \widetilde{v} = 0 \\ & (\widetilde{u}, w) \geq 0 \ , \end{aligned} \tag{3}$$

Using $C^\top = -C$, $E^\top = -E$ and eliminating $w \geq 0$ in the formulation, we get exactly (2) after replacing $\max(\cdot)$ by $\min -(\cdot)$.

3. Show that the LP (2) is equivalent to the linear complementarity problem (LCP):

$$\text{Find} \quad (u,v,w) \quad \text{such that} \quad \begin{cases} \begin{bmatrix} w \\ 0 \end{bmatrix} = \begin{bmatrix} C & D \\ -D^\top & E \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} f \\ g \end{bmatrix} \ , \\ (u,w) \geq 0 \ , \ u^\top w = 0 \ . \end{cases} \tag{4}$$

Solution: $\Rightarrow$ From the previous question, we deduce that any optimal solution of the dual is an optimal solution to the primal: $(u,v) = (\widetilde{u},\widetilde{v})$. Hence, we can eliminate the redundancy in the KKT equations explicited in the Question 1:

$$\begin{cases} f - C^\top u + D^\top v - w = 0 \\ g - D^\top u - E^\top v = 0 \\ u^\top (Cu + Dv + f) = 0 \\ (u,w) \geq 0 \end{cases}$$

Using the first equation, we get $w = Cu + Dv + f$. The complementarity condition reformulates as $u^\top w = 0$, and we deduce that (u,v,w) is solution of (4).

$\Leftarrow$ Suppose (u,v,w) is solution of (4). We can verify that $(u,v,\widetilde{u},\widetilde{v},w) := (u,v,u,v,w)$ is a primal-dual optimal solution of the LP (2).

4. Use the Goldman-Tucker theorem to prove that if the LP (2) is feasible, then the LCP (4) has a strictly complementary solution: i.e., there exists (u,v,w) solution of (4) such that $u + w > 0$.

Solution: Using the Goldman-Tucker theorem, there exists $(u,v,\widetilde{u},\widetilde{v},w)$ satisfying strict complementarity: $\widetilde{u} + (Cu + Dv + f) > 0$ and $u + w > 0$. Using self-duality, $(\widetilde{u},\widetilde{v},u,v,w)$ is also a solution satisfying strict complementarity, with non-zero products set at the same location as $(u,v,\widetilde{u},\widetilde{v},w)$. For $i = 1, \cdots, n$, we have either

$$(u_i, \widetilde{u}_i) > 0 \ , \ (Cu + Dv + f)_i = 0 \quad \text{or} \quad (u_i, \widetilde{u}_i) = 0 \ , \ (Cu + Dv + f)_i > 0 \ .$$

We deduce that for all i, $u_i + (Cu + Dv + f)_i > 0$, implying $u_i + w_i > 0$. As a consequence, (u,v,w) is strict complementarity solution of (4).

2 Simplified homogeneous self-dual (HSD) embedding

We introduce the following LP:

$$\begin{aligned} \min_{x,\lambda,\tau,s,\kappa} \quad & 0 \\ \text{s.t.} \quad & \begin{bmatrix} 0 & -A^\top & c \\ A & 0 & -b \\ -c^\top & b^\top & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \\ \tau \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ \kappa \end{bmatrix} \\ & (x, \tau, s, \kappa) \geq 0 \; . \end{aligned} \tag{5}$$

1. Show that the LP (5) has a trivial solution, and all feasible points are optimal and satisfy

$$s^\top x + \kappa\tau = 0 \; . \tag{6}$$

Solution: $(x, \lambda, \tau, s, \kappa) = (0, 0, 0, 0, 0)$ is a trivial solution. The objective function is 0, so all feasible points are optimal. For all feasible solution, we have:

$$\begin{aligned} s^\top x + \kappa\tau &= (-A^\top \lambda + \tau c)^\top x + (Ax - \tau b)^\top \lambda + (-c^\top x + b^\top \lambda)\tau \\ &= -\lambda^\top A x + \tau c^\top x + \lambda^\top A x - \tau b^\top \lambda - \tau c^\top x + \tau b^\top \lambda \\ &= 0 \end{aligned}$$

2. Justify that if $(x, \lambda, \tau, s, \kappa)$ is solution of the LP (5), then for all $t > 0$ the vector $(tx, t\lambda, t\tau, ts, t\kappa)$ is also a solution of (5).

Solution: The feasible set of LP (5) is an intersection of a linear subspace with a cone.

3. Suppose that $(x, \lambda, \tau, s, \kappa)$ is solution of (5), with $\tau > 0$ and $\kappa = 0$. Show that $(\hat{x}, \hat{\lambda}, \hat{s}) := (x/\tau, \lambda/\tau, s/\tau)$ is a solution of the original LP (1).

Solution: For $(x, \lambda, \tau, s, \kappa)$ solution of (5), $(\hat{x}, \hat{\lambda}, \hat{s})$ satisfies

$$\begin{cases} A^\top \hat{\lambda} - c + \hat{s} = 0 \\ A\hat{x} = b \\ c^\top \hat{x} = b^\top \hat{\lambda} \\ (\hat{x}, \hat{s}) \geq 0 \end{cases}$$

In addition, we have

$$\begin{aligned} \hat{s}^\top \hat{x} &= (-A^\top \hat{\lambda} + c)^\top \hat{x} \\ &= -\lambda^\top A\hat{x} + c^\top \hat{x} \\ &= -\lambda^\top A\hat{x} + \hat{\lambda}^\top b \end{aligned}$$

As $A\hat{x} = b$, we deduce $\hat{s}^\top \hat{x} = 0$. Hence, $(\hat{x}, \hat{\lambda}, \hat{s})$ satisfies the KKT conditions of the LP (1):

$$\begin{cases} c - A^\top \lambda - s = 0 \\ Ax = b \\ s^\top x = 0 \\ (x, s) \geq 0 \end{cases}$$

4. Show that the LP (5) has the same structure as the self-dual LP (2). Explicit the matrix C, D, E and the vectors f, g. Deduce that the original LP (2) is equivalent to the LCP:

$$\text{Find} \quad (x,\lambda,\tau,s,\kappa) \quad \text{such that} \quad \begin{cases} \begin{bmatrix} 0 & -A^\top & c \\ A & 0 & -b \\ -c^\top & b^\top & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \\ \tau \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ \kappa \end{bmatrix} \\ (x,\tau,s,\kappa) \geq 0 \;, \quad s^\top x + \kappa\tau = 0 \;. \end{cases} \tag{7}$$

Solution: We identify the vectors and the matrices one by one. First, we have $f = 0$ and $g = 0$ as the objective is zero. Second, we have $C = \begin{bmatrix} 0 & -A^\top \\ A & 0 \end{bmatrix}$ skew-symmetric, $E = 0$ and $D = \begin{bmatrix} c \\ -b \end{bmatrix}$. The equivalence with (7) follows from Section 1

3 Generalization to conic programming

We recall the following result.

Theorem 2 (Moreau decomposition)**.** *Let K be a closed convex cone and $K^\oplus$ its positive dual cone. We note P_K (resp. $P_{-K^\oplus}$) the Euclidean projection onto K (resp. $-K^\oplus$). For $x, y, z \in \mathbb{R}^n$, the following statements are equivalent:*

1. *$z = x + y$ for $x \in K$, $y \in -K^\oplus$ and $x^\top y = 0$.*
2. *$x = P_K(z)$ and $y = P_{-K^\oplus}(z)$.*

1. Let $K = \mathbb{R}^n_+ \times \mathbb{R}^m \times \mathbb{R}_+$.

   (a) Give the expression of the dual cone $K^\oplus$.

   Solution: The positive orthant $\mathbb{R}^n_+$ is self-dual, and $(\mathbb{R}^m)^\oplus = \{0\}^m$. Hence $K^\oplus = \mathbb{R}^n_+ \times \{0\}^m \times \mathbb{R}_+$.

   (b) For a matrix Q to be explicited, deduce that we can rewrite (7) more compactly as

$$\text{Find} \quad (u,v) \quad \text{such that} \quad \begin{cases} v = Qu \;, \\ (u,v) \in K \times K^\oplus \;. \end{cases} \tag{8}$$

   Solution: Let $Q = \begin{bmatrix} 0 & -A^\top & c \\ A & 0 & -b \\ -c^\top & b^\top & 0 \end{bmatrix}$. For $(x,\lambda,\tau,s,\kappa)$ solution of (7), we have $u := (x,\lambda,\tau) \in K$ and $v = (s,0,\kappa) \in K^\oplus$ and $v = Qu$. Conversely, we can build a solution of (7) from (u,v) solution of (8).

   (c) Use the Moreau decomposition theorem to prove that (8) is equivalent to finding $z \in \mathbb{R}^{n+m+1}$ such that

$$-P_{-K^\oplus}(z) = QP_K(z) \;. \tag{9}$$

Solution: The proof is direct using the Moreau decomposition theorem.

2. **[Bonus question:]** Let $\mathcal{C} \subset \mathbb{R}^n$ a proper cone. How to adapt the homogeneous self-dual embedding to solve the following conic problem?

$$\min_{x \in \mathbb{R}^n} c^\top x \quad \text{s.t.} \quad Ax = b\ ,\ x \in \mathcal{C}\ . \tag{10}$$

Solution: The KKT conditions of the conic program are

$$\begin{cases} c - A^\top \lambda - s = 0 \\ Ax = b \\ s^\top x = 0 \\ (x, s) \in \mathcal{C} \times \mathcal{C}^\oplus \end{cases} \tag{11}$$

Using the same procedure as before, we can prove that the KKT system is equivalent to the HSD embedding (8), with $K = \mathcal{C} \times \mathbb{R}^m \times \mathbb{R}_+$ and $K^\oplus = \mathcal{C}^\oplus \times \{0\}^m \times \mathbb{R}_+$.