Take home exam 2022

April 14, 2022

Personal work, to be returned on May 13th. Estimated work time : 4-6 hours.

Exercise 1 (Decomposition theorem)**.** *Let K be a non-empty, closed, convex subset of $\mathbb{R}^n$. We denote $K^\ominus = -K^\oplus = \{y \in \mathbb{R}^n \quad | \quad \langle y, z\rangle \leq 0, \quad \forall z \in K\}$.*
Let $x \in \mathbb{R}^n$.

1. *Show that there exists a unique $y \in K$, called the* projection of x on K, *and denoted* $\mathrm{proj}_K(x)$ *such that*
$$\|x - y\|_2 = \inf_{z \in K} \|x - z\|_2$$
2. *Show that $y = \mathrm{proj}_K(x)$ is the only element of K such that*
$$\langle x - y, z - y\rangle \leq 0, \qquad \forall z \in K$$
3. *From now on, assume that K is a closed convex cone. Show that there exists $y \in K$, and $z \in K^\ominus$, with $\langle y, z\rangle = 0$, such that $x = y + z$.*
4. *Deduce from the previous point that $K^{\ominus\ominus} = K$.*
5. *Show that we have $x = y + z$, with $y \in K$, and $z \in K^\ominus$, $\langle y, z\rangle = 0$, if and only if $y = \mathrm{proj}_K(x)$ and $z = \mathrm{proj}_{K^\ominus}(x)$.*

Exercise 2 (SOCP)**.** *We define the second order cone $K_n = \{(x,t) \in \mathbb{R}^{n+1} \quad | \quad t \geq \|x\|_2\}$. We say that a constraint is* second order cone (SOC) representable *if it can be written as $(y,\theta) \in K_m$ for adequately chosen y and θ.*
We say that an optimization problem is an SOCP in standard form if it is written as

$$\underset{x \in \mathbb{R}^n}{\mathrm{Min}} \quad c_0^\top x \tag{1a}$$
$$s.t. \quad \|A_i^\top x + b_i\| \leq c_i^\top x + d_i \qquad \forall i \in [n] \tag{1b}$$

1. *Show that $\{w^T w \leq xy, x \geq 0, y \geq 0\}$ is equivalent to $\left\| \begin{pmatrix} 2w \\ x - y \end{pmatrix} \right\| \leq x + y$ where $w \in \mathbb{R}^n, x \in \mathbb{R}$, and $y \in \mathbb{R}$. Deduce that $\{w \in \mathbb{R}^n, x \geq 0, y \geq 0 \mid w^T w \leq xy\}$ is SOC representable.*
2. *Show that, for any matrix and vector of adequate dimension, $\{x \in \mathbb{R}^n \mid \|Ax + b\|_2 \leq c^\top x + d\}$ is SOC representable.*
3. *Show that, for any $Q \in S_n^{++}$, the constraint set $\{(x,t) \in \mathbb{R}^{n+1} \mid x^\top Q x \leq t\}$ is SOC representable.*

4. Represent the following LP program as an SOCP in standard form.

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{Min}} \quad & c_0^\top x \\ s.t. \quad & Ax \leq b \end{aligned}$$

5. Represent the following convex QP program as an SOCP in standard form.

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{Min}} \quad & x^\top Q x + c_0^\top x \\ s.t. \quad & Ax \leq b \end{aligned}$$

6. Compute, for $v \in \mathbb{R}^n$ *and* $\lambda > 0$, $\sup \left\{ u^\top v \mid \|u\|_2 \leq \lambda \right\}$.

7. Show that the SOCP problem in standard form (1) *admit the following dual formulation [hint: you can use the results of the previous question]*

$$\begin{aligned} \underset{(u_i,\mu_i)_{i\in[m]} \in (\mathbb{R}^n \times \mathbb{R}_+)^m}{\text{Max}} \quad & \sum_{i=1}^m u_i^\top b_i - \mu_i d_i \\ s.t. \quad & \sum_{i=1}^m (A_i^\top u_i - \mu_i c_i) = c_0 \\ & \|u_i\|_2 \leq \mu_i \qquad\qquad i \in [m] \end{aligned}$$

Is it an SOCP ? Give a simple condition to have strong duality.

8. Represent the following square root lasso problem

$$\underset{w \in \mathbb{R}^n}{\text{Min}} \quad \|Aw - b\| + r\|w\|_1$$

as an SOCP, and give a dual formulation. Is there a duality gap ?