Take home exam 2022

May 13, 2022

Personal work, to be returned on May 13th. Estimated work time : 4-6 hours.

Exercise 1 (Decomposition theorem)**.** *Let K be a non-empty, closed, convex subset of $\mathbb{R}^n$. We denote $K^{\ominus} = -K^{\oplus} = \big\{y \in \mathbb{R}^n \quad | \quad \big\langle y\,, z\big\rangle \leq 0, \quad \forall z \in K\big\}$.*
Let $x \in \mathbb{R}^n$.

1. *Show that there exists a unique $y \in K$, called the* projection of x on K, *and denoted* $\mathrm{proj}_K(x)$ *such that*
$$\|x - y\|_2 = \inf_{z \in K} \|x - z\|_2$$
2. *Show that $y = \mathrm{proj}_K(x)$ is the only element of K such that*
$$\big\langle x - y\,, z - y\big\rangle \leq 0, \qquad \forall z \in K$$
3. *From now on, assume that K is a closed convex cone. Show that there exists $y \in K$, and $z \in K^{\ominus}$, with $\big\langle y\,, z\big\rangle = 0$, such that $x = y + z$.*
4. *Deduce from the previous point that $K^{\ominus\ominus} = K$.*
5. *Show that we have $x = y + z$, with $y \in K$, and $z \in K^{\ominus}$, $\big\langle y\,, z\big\rangle = 0$, if and only if $y = \mathrm{proj}_K(x)$ and $z = \mathrm{proj}_{K^{\ominus}}(x)$.*

Solution.

1. *$\mathrm{proj}_K(x)$ is an optimal solution to the following optimization problem : $\mathrm{Min}_{z \in K} f(z) := \|z - x\|_2^2$, which is convex with strongly convex objective function, ensuring existence (0.5pts) and unicity (0.5pts).*
2. *The convex optimality condition reads $-\nabla f(y) \in N_K(y)$, which yields $\big\langle -(y-x)\,, z-y\big\rangle \leq 0$.(1pts)*
3. *(2pts)We set $y = \mathrm{proj}_K(x)$, and $z = x - y$. Thus, for all $p \in K$, $\big\langle x - y\,, p - y\big\rangle \leq 0$. We choose $p = ty$, for $t > 0$. It follows, $(t-1)\big\langle z\,, y\big\rangle \leq 0$, thus $\big\langle z\,, y\big\rangle = 0$.*

   Finally, as $0 \geq \big\langle x - y\,, p - y\big\rangle = \big\langle z\,, p\big\rangle - \big\langle z\,, y\big\rangle = \big\langle z\,, p\big\rangle$ we have $z \in K^{\ominus}$.
4. *(2pts) Obviously $K \subset K^{\ominus\ominus}$. Consider $x \in K^{\ominus\ominus}$, and $y \in K$, $z \in K^{\ominus}$ such that $x = y + z$ and $\big\langle y\,, z\big\rangle = 0$. Then,*
$$0 \geq \big\langle x\,, z\big\rangle = \big\langle y\,, z\big\rangle + \big\langle z\,, z\big\rangle = \|z\|^2$$

   Which means that $z = 0$ and $x = y$.

5. (2pts) *The only if part is straight from the proof of 3. Now consider y and z satisfying the conditions. Then, for any $p \in K$ we have*

$$\langle z - x \,, p - x \rangle = \langle y \,, p - x \rangle = \langle y \,, p \rangle \leq 0,$$

which characterize the projection on K by question 2. We can do the same for the projection on $K^{\ominus}$.

Exercise 2 (SOCP)**.** *We define the second order cone $K_n = \{(x,t) \in \mathbb{R}^{n+1} \quad | \quad t \geq \|x\|_2\}$. We say that a constraint is* second order cone (SOC) representable *if it can be written as $(y,\theta) \in K_m$ for adequately chosen y and θ.*
We say that an optimization problem is an SOCP in standard form if it is written as

$$\underset{x\in\mathbb{R}^n}{\text{Min}} \quad c_0^\top x \tag{1a}$$

$$s.t. \quad \|A_i^\top x + b_i\| \leq c_i^\top x + d_i \qquad \forall i \in [n] \tag{1b}$$

1. *Show that $\{w^T w \leq xy, x \geq 0, y \geq 0\}$ is equivalent to $\left\|\begin{pmatrix} 2w \\ x-y \end{pmatrix}\right\| \leq x+y$ where $w \in \mathbb{R}^n, x \in \mathbb{R}$, and $y \in \mathbb{R}$. Deduce that $\{w \in \mathbb{R}^n, x \geq 0, y \geq 0 \mid w^T w \leq xy\}$ is SOC representable.*

2. *Show that, for any matrix and vector of adequate dimension, $\{x \in \mathbb{R}^n \mid \|Ax + b\|_2 \leq c^\top x + d\}$ is SOC representable.*

3. *Show that, for any $Q \in S_n^{++}$, the constraint set $\{(x,t) \in \mathbb{R}^{n+1} \mid x^\top Q x \leq t\}$ is SOC representable.*

4. *Represent the following LP program as an SOCP in standard form.*

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{Min}} \quad & c_0^\top x \\ s.t. \quad & Ax \leq b \end{aligned}$$

5. *Represent the following convex QP program as an SOCP in standard form.*

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{Min}} \quad & x^\top Q x + c_0^\top x \\ s.t. \quad & Ax \leq b \end{aligned}$$

6. *Compute, for $v \in \mathbb{R}^n$ and $\lambda > 0$, $\sup\{u^\top v \mid \|u\|_2 \leq \lambda\}$.*

7. *Show that the SOCP problem in standard form (1) admit the following dual formulation [hint: you can use the results of the previous question]*

$$\begin{aligned} \underset{(u_i,\mu_i)_{i\in[m]}\in(\mathbb{R}^n\times\mathbb{R}_+)^m}{\text{Max}} \quad & \sum_{i=1}^m u_i^\top b_i - \mu_i d_i \\ s.t. \quad & \sum_{i=1}^m (A_i^\top u_i - \mu_i c_i) = c_0 \\ & \|u_i\|_2 \leq \mu_i \qquad\qquad i \in [m] \end{aligned}$$

Is it an SOCP ? Give a simple condition to have strong duality.

8. *Represent the following square root lasso problem*

$$\operatorname*{Min}_{w\in\mathbb{R}^n} \quad \|Aw-b\| + r\|w\|_1$$

as an SOCP, and give a dual formulation. Is there a duality gap ?

Solution. 1. (1pts) *Taking the square of the norm (possible by non-negativity) we get* $4w^\top w + 2(x-y)^2 \leq (x+y)^2$ *developing and checking sign yields the result.*

2. (0.5pts) $y = Ax+b$ *and* $\theta = c^\top x + d$.

3. (1pts) $w = Q^{1/2}x$, $\theta = t$, *we then have* $w^\top w \leq t1$ *and the previous reformulation yields*

$$\left\| \begin{pmatrix} 2Q^{1/2}x \\ t-1 \end{pmatrix} \right\| \leq t+1$$

4. (0.5pts) $C_i \leftarrow 0$, $b_i \leftarrow 0$, $c_i \leftarrow -a_i$ *and* $d_i \leftarrow b_i$

5. (0.5pts)

$$\begin{aligned} \operatorname*{Min}_{x\in\mathbb{R}^n, t\geq 0} \quad & t + c_0^\top x \\ s.t. \quad & Ax \leq b \\ & \left\| \begin{pmatrix} 2Q^{1/2}x \\ t-1 \end{pmatrix} \right\| \leq t+1 \end{aligned}$$

6. (0.5pts) $\sup\left\{u^\top v \mid \|u\|_2 \leq \lambda\right\} = \lambda\|v\|_2$

7. (3pts) *We have*

$$\begin{aligned} p^\sharp &= \operatorname*{Min}_{x\in\mathbb{R}^n} \quad c_0^\top x + \sum_{i=1}^m \sup_{\mu_i\geq 0} \mu_i\Big[\|A_i x + b_i\|_2 - (c_i^\top x + d_i)\Big] \\ &= \operatorname*{Min}_{x\in\mathbb{R}^n} \quad c_0^\top x + \sum_{i=1}^m \sup_{\mu_i\geq 0}\Big(\sup_{u_i:\|u_i\|\leq\mu_i} u_i^\top(A_i x + b_i) - \mu_i(c_i^\top x + d_i)\Big) \end{aligned}$$

And the dual reads

$$\begin{aligned} d^\sharp = \operatorname*{Max}_{(u_i,\mu_i)_{i\in[m]}\in(\mathbb{R}^n\times\mathbb{R}_+)^m} \quad & \inf_{x\in\mathbb{R}^n} c_0^\top x + \sum_{i=1}^m \Big(u_i^\top(A_i x + b_i) - \mu_i(c_i^\top x + d_i)\Big) \\ s.t. \quad & \|u_i\| \leq \mu_i \\ = \operatorname*{Max}_{(u_i,\mu_i)_{i\in[m]}\in(\mathbb{R}^n\times\mathbb{R}_+)^m} \quad & \sum_{i=1}^m u_i^\top b_i - \mu_i^\top d_i + \inf_{x\in\mathbb{R}^n} x^\top\Big(c + \sum_{i=1}^m A_i^\top u_i - \mu_i c_i\Big) \\ s.t. \quad & \|u_i\| \leq \mu_i \end{aligned}$$

8. (2pts)