Convex Optimization Exam

16/06/2023
3 hours – documents allowed
Answers in English or French

The exam is made of 4 independent exercises, in roughly increasing difficulty. If necessary, you can admit the results of previous questions. When using the recalls, cite them. "Classifying" an optimization problem consists in precising in which of the category presented in chapter 5 it falls (LP, QP, QCQP, SOCP, SDP, unconstrained or not, differentiable or not, continuous or not, convex or not).

Some usefull recalls

i) The infimal convolution of f and g is defined as $f \square g(x) = \inf_{y\in\mathbb{R}^n} f(y) + g(x-y)$.

ii) The Fenchel transform of a function f is defined as $f^\star : x^\star \mapsto \sup_{x\in\mathbb{R}^n} \langle x^\star, x\rangle - f(x)$.

Exercice 1: Breakfast *4 points*

We consider the following problem

$$\begin{aligned} \underset{x\in\mathbb{R}}{\text{Min}} \quad & (x-1)^2 \\ \text{s.t.} \quad & x \leq 0 \end{aligned}$$

In this exercise, x_+ denotes the positive part of x, i.e. $x_+ = \max(x, 0)$.

(a) (1 point) Solve the problem using the KKT conditions. What is the optimal primal and dual solution?

(b) (1 point) We now consider the penalized problem

$$(\mathcal{P}_t^2) \qquad \underset{x\in\mathbb{R}}{\text{Min}} \quad (x-1)^2 + t(x_+)^2.$$

Find the optimal solution $x_t^{(2)}$ for $t > 0$. What can we observe about $t \mapsto x_t^{(2)}$?

(c) (1 point) We now consider the penalized problem

$$(\mathcal{P}_t^1) \qquad \underset{x\in\mathbb{R}}{\text{Min}} \quad (x-1)^2 + t(x_+).$$

Find the optimal solution $x_t^{(1)}$ for $t > 0$. What can we observe about $t \mapsto x_t^{(1)}$?

(d) (1 point) We now consider the penalized problem

$$(\mathcal{P}_s^{log}) \qquad \underset{x\in\mathbb{R}}{\text{Min}} \quad (x-1)^2 - s\log(-x).$$

Find the optimal solution $x_s^{(log)}$ for $s > 0$. What can we observe about $s \mapsto x_s^{(log)}$?

Solution:

(a) TODO

(b) $x_t^{(2)} = \dfrac{1}{1+t}$, we see that $x_t^{(2)} \to x^\sharp$ when $t \to +\infty$, and $x_t^{(2)} > 0$ for all $t > 0$.

(c) $x_t^{(1)} = (1 - t/2)_+$, we see that $x_t^{(1)} = x^\sharp$ when $t \geq 2$.

(d) $x_s^{(\log)} = \frac{1-\sqrt{1+2s}}{2}$, we see that $x_s^{(\log)} \to x^\sharp$ when $s \to 0$, and $x_s^{(\log)} < 0$ for all $s > 0$.

Exercice 2: Warm-up *4 points*

(a) (2 points) We consider the following optimization problem

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{Min}} \quad & c^\top x \\ \text{s.t.} \quad & x^\top A x \leq 1 \end{aligned}$$

for some symmetric matrix A and some vector $c \in \mathbb{R}^n$.

i) Classify this problem.

ii) Solve the problem assuming that $A \succ 0$.

iii) Solve it assuming that $A \not\succeq 0$.

(b) (2 points) Let f and g be convex function of $\mathbb{R}^n$ to $\mathbb{R}$.

i) Show that $f \Box g$ is convex.

ii) Show that $(f \Box g)^\star = f^\star + g^\star$

Solution:

(a) i) It is a QCQP, convex iff $Q \succeq 0$.
ii) If $A \succ 0$, then we set $y = Q^{1/2}x$ and the problem is equivalent to

$$\begin{aligned} \underset{y\in\mathbb{R}^n}{\text{Min}} \quad & (Q^{-1/2}c)^\top y \\ \text{s.t.} \quad & \|y\|_2 \leq 1 \end{aligned}$$

with optimal solution $-\tilde{c}/\|\tilde{c}\|_2$ for $\tilde{c} = Q^{-1/2}c$.
iii) If $A \not\succeq 0$, then the problem is unbounded below.

(b) i) partial infimum of convex functions is convex.
ii)

$$\begin{aligned} (f\Box g)^\star(x^\star) &= \sup_{x\in\mathbb{R}^n} \langle x^\star, x\rangle - f\Box g(x) \\ &= \sup_{x\in\mathbb{R}^n} \langle x^\star, x\rangle - \inf_{y\in\mathbb{R}^n} f(y) + g(x-y) \\ &= \sup_{x\in\mathbb{R}^n} \sup_{y\in\mathbb{R}^n} \langle x^\star, x\rangle - f(y) - g(x-y) \\ &= \sup_{x\in\mathbb{R}^n} \sup_{y\in\mathbb{R}^n} \langle x^\star, y\rangle + \langle x^\star, x-y\rangle - f(y) - g(x-y) \\ &= \sup_{x\in\mathbb{R}^n} \langle x^\star, y\rangle - f(y) + \sup_{z\in\mathbb{R}^n} \langle x^\star, z\rangle - g(z) & z = x-y \\ &= f^\star(x^\star) + g^\star(x^\star) \end{aligned}$$

Exercice 3: Bifurcation de solution *4 points*

We consider, for $\varepsilon \in \mathbb{R}$, the problem

$$\begin{aligned} (\mathcal{P}_\varepsilon) \qquad \underset{x\in\mathbb{R}^2}{\text{Min}} \quad & x_2^2 - x_1^2 & \text{(1a)} \\ \text{s.t.} \quad & x_2 \geq 2|x_1| - \varepsilon & \text{(1b)} \end{aligned}$$

(a) (1 point) Classify the problem. Reformulate the problem's constraint as two linear constraints.

(b) (1 point) Show the existence of optimal solutions for all ε.

(c) (2 points) For all ε find the stationary points (i.e. the points satisfying the KKT conditions) of $(\mathcal{P}_\varepsilon)$. Plot them in the (x_1, ε) and (x_2, ε) plane (two plots).

Exercice 4: Logistic regression and exponential cone *8 points*

We recall the logistic regression problem:

$$\min_{\theta \in \mathbb{R}^n} \; L(\theta) \quad \text{with} \quad L(\theta) := \frac{1}{m} \sum_{i=1}^{m} \log\left(1 + \exp(-y_i \cdot \theta^\top x_i)\right) . \tag{2}$$

We define the exponential cone as

$$K_{exp} = \mathrm{cl}\{x \in \mathbb{R}^3 \mid x_1 \geq x_2 \exp(x_3/x_2),\ x_2 > 0\} , \tag{3}$$

with cl the closure operator. We admit that K_{exp} is closed convex cone.

(a) (1 point) Let $x, t \in \mathbb{R}$. The constraint $\exp(x) \leq t$ is equivalent to the conic representation $(t, 1, x) \in K_{exp}$. Similarly, find the conic representation of

  i) the log constraint $\log(x) \geq t$,

  ii) the entropic constraint $-x \log(x) \geq t$.

As we have seen, the exponential cone is a powerful tool to find a conic representation for constraints involving exponential and logarithm terms, both presents in the logistic regression problem (2). Using variable lifting, the logistic regression problem (2) is equivalent to

$$\min_{\theta \in \mathbb{R}^n, t \in \mathbb{R}^m} \quad \sum_{i=1}^{m} t_i \tag{4a}$$

$$\text{s.t.} \quad t_i \geq \log(1 + \exp(-y_i \cdot \theta^\top x_i)) \qquad \forall i \in [m] . \tag{4b}$$

(b) (2 points) Let us consider the softplus constraint $\log(1 + \exp(x)) \leq t$, for $x, t \in \mathbb{R}$. Show that it is equivalent to

$$\begin{cases} \exp(x - t) \leq u , \\ \exp(-t) \leq v , \\ u + v \leq 1 . \end{cases} \tag{5}$$

Find a conic representation of (5) with one affine constraint and two conic constraints involving the exponential cone K_{exp}.

(c) (1 point) Use the previous question to find a conic representation for the logistic regression problem (4). Discuss (i) the number of variables, and (ii) the number of constraints.

(d) (1 point) Solving a conic optimization problem requires an efficient projection operator onto the given convex cone K. Let $x \in \mathbb{R}^3$. Shows there is a unique $y \in K_{exp}$ (and denoted by $\mathrm{proj}_{K_{exp}}(x)$) such that

$$y \in \arg\min_{z \in K_{exp}} \|x - z\|_2 . \tag{6}$$

(e) (1 point) Show that $y = \mathrm{proj}_{K_{exp}}(x)$ if and only if $\langle x - y, z - y \rangle \leq 0$ for all $z \in K_{exp}$.

(f) (2 points) The Moreau decomposition theorem (admitted) shows that every $x \in \mathbb{R}^3$ can be decomposed as

$$x = y + z , \quad y \in K_{exp} , \quad z \in K_{exp}^- , \quad y^\top z = 0 , \tag{7}$$

where K_{exp}^- is the polar form of the exponential cone K_{exp}. Shows that (7) are exactly the KKT conditions of the projection problem (6).