Introduction to Optimization

Vincent Leclère (CERMICS, ENPC)

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Why should I bother to learn this stuff?

- Many engineering problems are naturally linear: production planning, blending, flows, transportation, energy dispatch, ...
- Linear programs with very large numbers of variables and constraints can be solved routinely by modern software.
- The simplex method explains one of the fundamental mechanisms behind industrial optimization solvers.
- LP duality also gives economically meaningful information such as resource values and sensitivities.

Contents

Linear programming

A linear program is an optimization problem of the form

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{Min}} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b \\ & Ex = d \end{aligned}$$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $E \in \mathbb{R}^{p\times n}$ and $d \in \mathbb{R}^p$ are given data.

- The objective function is linear: $f(x) = c^\top x$.
- The constraints are linear inequalities or equalities.
- The feasible set is a polyhedron (intersection of half-spaces and hyperplanes).
- Linear programs can be solved efficiently with dedicated algorithms (simplex, interior-point methods, ...).

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Linear programs as convex optimization problems

A linear program is a convex optimization problem with a linear objective function and a polyhedral feasible set.

The objective function $f(x) = c^\top x$ is convex and differentiable, with $\nabla f(x) = c$.

The feasible set

$$X = \{x \in \mathbb{R}^n \mid Ax \leq b, \; Ex = d\}$$

is a closed convex polyhedron.

Therefore, convex optimization theory applies:

- constraints are qualified at every feasible point;
- KKT conditions are necessary and sufficient for optimality;
- strong duality holds[1];
- the dual of a linear program is also a linear program.

[1] If the primal or the dual is feasible.

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Linear programs: inequality and standard forms

Two useful formulations are:

- Inequality form: convenient for geometric interpretation

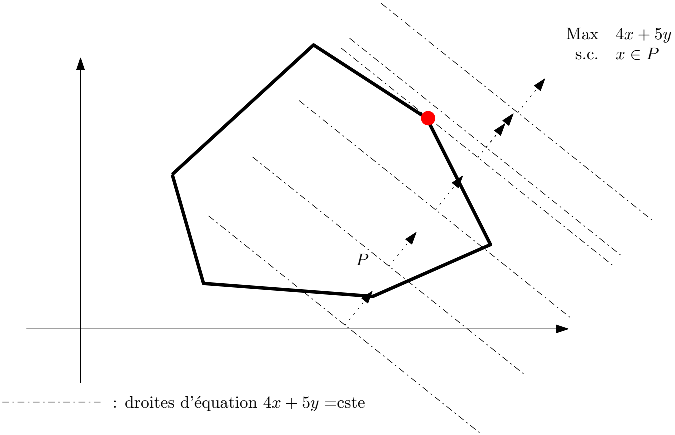
$$
\begin{aligned}
\underset{x \in \mathbb{R}^n}{\text{Min}} \quad & c^\top x \\
\text{s.t.} \quad & Ax \leq b.
\end{aligned}
$$

- Standard form: convenient for the simplex method

$$
\begin{aligned}
\underset{x \in \mathbb{R}^n}{\text{Min}} \quad & c^\top x \\
\text{s.t.} \quad & Ax = b \\
& x \geq 0.
\end{aligned}
$$

Exercise. Show that any linear program can be converted to either form.

Geometric interpretation – recall



- The feasible set is a polyhedron.
- If the problem is feasible and bounded below, an optimal solution exists.
- When the feasible polyhedron has vertices, an optimum can be chosen at a vertex.

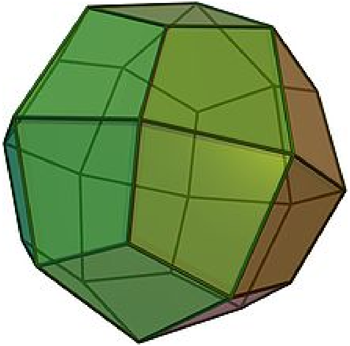
Contents

Definitions

- A half-space is a set of the form $\{x \in \mathbb{R}^n \mid a^\top x \leq b\}$, with $a \in \mathbb{R}^n$, $b \in \mathbb{R}$.
- A polyhedron is an intersection of finitely many half-spaces.

➥ A polyhedron is convex.

- A polytope is a bounded polyhedron.
- A face of a polyhedron P is the intersection of P with a supporting hyperplane.
- A vertex (or extreme point) is a face of dimension 0.
- An edge is a face of dimension 1.



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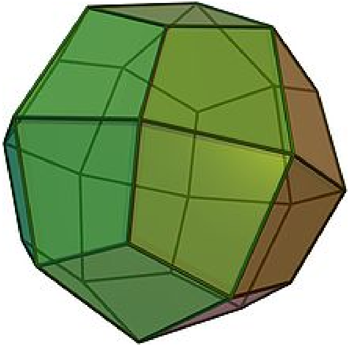


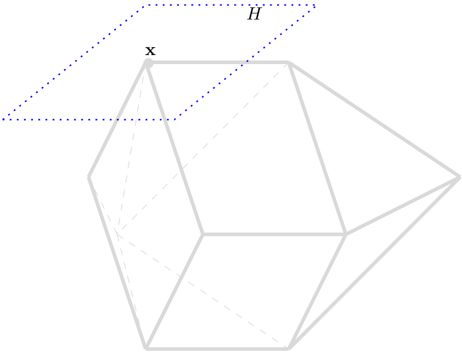
Vertex

Definition (Vertex)

A point $\boldsymbol{x} \in P$ is a vertex of the polyhedron P if there exists a vector $c \in \mathbb{R}^n$ such that $\boldsymbol{x}$ is the unique solution of

$$\underset{\boldsymbol{y} \in P}{\text{Min}} \quad c^\top \boldsymbol{y}.$$

Equivalently, $\boldsymbol{x} \in P$ is a vertex if it cannot be expressed as a strict convex combination of two distinct points of P.



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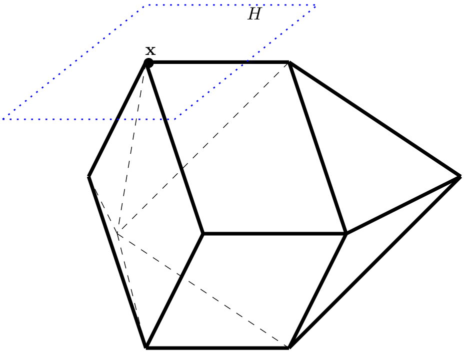

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Standard form

We consider a linear program in standard form:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{Min}} \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ are given data.
We assume that

- A has full row rank: $\mathrm{rank}(A) = m$;
- $m < n$ (otherwise the solution is trivial);
- the problem is feasible: there exists x such that $Ax = b$, $x \geq 0$.

For the moment, we also assume that an initial feasible basis is available. We will return to this point later.

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Some notation

Let $B \subset [n]$ with $|B| = m$. We denote by

- $A_B \in \mathbb{R}^{m\times m}$ the matrix composed of the columns of A indexed by B;
- $x_B \in \mathbb{R}^m$ the vector composed of the components of x indexed by B;
- $c_B \in \mathbb{R}^m$ the vector composed of the components of c indexed by B.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 0 & 5 \end{pmatrix} \quad \boldsymbol{x} = \begin{pmatrix} 2 \\ 9 \\ 4 \end{pmatrix} \quad B = \{1, 3\}$$

$$A_B = ? \qquad \boldsymbol{x}_B = ?$$

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Basis and basic solution

$$A\boldsymbol{x} = \boldsymbol{b} \iff A_B\boldsymbol{x}_B + A_N\boldsymbol{x}_N = \boldsymbol{b} \iff \boldsymbol{x}_B = A_B^{-1}\boldsymbol{b} - A_B^{-1}A_N\boldsymbol{x}_N.$$

Definition (Basis and basic solution)

A set $B \subset [n]$ of size m such that A_B is invertible is called a basis.
It is a feasible basis if $A_B^{-1}b \geq 0$.
The vector x defined by

$$\begin{aligned} x_B &= A_B^{-1}b, \\ x_{\underbrace{[n] \setminus B}_{N}} &= 0, \end{aligned}$$

is the basic solution associated with B; it is a basic feasible solution (BFS) when B is feasible.

A basic solution is not necessarily feasible: some components of $A_B^{-1}b$ may be negative.

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Basic feasible solution and vertex

Theorem

A point $\boldsymbol{x}$ is a vertex of the feasible set if and only if it is a basic feasible solution.

Indeed:

- Let $\boldsymbol{x} = (\boldsymbol{x}_B, \boldsymbol{x}_N)$ be a basic feasible solution. Consider $\boldsymbol{y}, \boldsymbol{z} \in X$, and $\alpha \in (0,1)$ such that

  $$\boldsymbol{x} = \alpha \boldsymbol{y} + (1-\alpha)\boldsymbol{z}.$$

  Since $\boldsymbol{x}_N = 0$ and $\boldsymbol{y}, \boldsymbol{z} \geq 0$, we have $\boldsymbol{y}_N = \boldsymbol{z}_N = 0$. Therefore, $A_B \boldsymbol{x}_B = b = A_B \boldsymbol{y}_B = A_B \boldsymbol{z}_B$. Since A_B is invertible, $\boldsymbol{x}_B = \boldsymbol{y}_B = \boldsymbol{z}_B$, hence $\boldsymbol{y} = \boldsymbol{z} = \boldsymbol{x}$ and $\boldsymbol{x}$ is a vertex.

- Conversely, let $\boldsymbol{x}$ be a vertex of X.
  - If $\{A_j \mid j \in \operatorname{supp}(\boldsymbol{x})\}$ were linearly dependent, there would exist a non-zero $\boldsymbol{d}$ such that $A\boldsymbol{d} = 0$ and $d_j = 0$ outside $\operatorname{supp}(\boldsymbol{x})$.
  - For $\varepsilon > 0$ small enough, $\boldsymbol{x} \pm \varepsilon \boldsymbol{d}$ would both be feasible, contradicting extremality.
  - Therefore $\{A_j \mid j \in \operatorname{supp}(\boldsymbol{x})\}$ is linearly independent and can be completed to a basis B. $\rightsquigarrow$ Multiple bases may correspond to the same vertex.

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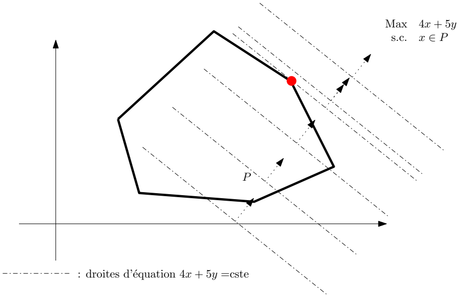
Existence of an optimal solution

Theorem (Fundamental theorem of linear programming)

If a linear program is feasible and bounded below, then it has an optimal solution.
If a linear program has an optimal solution, then it has an optimal basic feasible solution, or equivalently an optimal solution that is a vertex of the feasible set.

If several solutions are optimal, every convex combination of them is also optimal.



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Simplex method philosophy

♡

The simplex method is a popular algorithm for solving linear programs.

It works by moving from one vertex of the feasible set to another, while decreasing the objective function value at each iteration, until an optimal vertex is reached.

Moving from one vertex to another means adding one index (the entering index) to the basis B and removing another one (the leaving index), so that the new basis $\widetilde{B}$ is still feasible.

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Reduced cost

Let B be a basis. We have

$$A\boldsymbol{x} = \boldsymbol{b} \iff A_B\boldsymbol{x}_B + A_N\boldsymbol{x}_N = \boldsymbol{b} \iff \boldsymbol{x}_B = A_B^{-1}\boldsymbol{b} - A_B^{-1}A_N\boldsymbol{x}_N.$$

Thus the standard-form linear program can be rewritten as

$$\begin{aligned} \underset{x_N\in\mathbb{R}^{n-m}}{\text{Min}} \quad & \\ \text{s.t.} \quad & A_B^{-1}\boldsymbol{b} - A_B^{-1}A_N x_N \geq 0 \\ & x_N \geq 0. \end{aligned}$$

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$$\begin{aligned} \underset{x_N \in \mathbb{R}^{n-m}}{\text{Min}} \quad & \boldsymbol{c}_B^\top A_B^{-1}\boldsymbol{b} + (\boldsymbol{c}_N^\top - \boldsymbol{c}_B^\top A_B^{-1}A_N)x_N \\ \text{s.t.} \quad & A_B^{-1}\boldsymbol{b} - A_B^{-1}A_N x_N \geq 0 \\ & x_N \geq 0. \end{aligned}$$

The vector

$$\boldsymbol{r}_B^\top = \boldsymbol{c}_N^\top - \boldsymbol{c}_B^\top A_B^{-1}A_N$$

is the reduced cost vector associated with the basis B.

Optimality condition

Theorem (Optimality condition)

Let B be a basis such that the associated basic solution $\check{x}$ is feasible.
If the associated reduced cost vector $\boldsymbol{r}_B$ satisfies $\boldsymbol{r}_B \geq 0$, then $\check{x}$ is an optimal solution of the linear program.

Indeed, for any feasible $\boldsymbol{x}$,

$$\boldsymbol{x}_B = A_B^{-1}\boldsymbol{b} - A_B^{-1}A_N\boldsymbol{x}_N,$$

hence

$$\begin{aligned}\boldsymbol{c}^\top\boldsymbol{x} &= \boldsymbol{c}_B^\top A_B^{-1}\boldsymbol{b} + (\boldsymbol{c}_N^\top - \boldsymbol{c}_B^\top A_B^{-1}A_N)\boldsymbol{x}_N \\ &= \boldsymbol{c}^\top\check{\boldsymbol{x}} + \boldsymbol{r}_B^\top\boldsymbol{x}_N \geq \boldsymbol{c}^\top\check{\boldsymbol{x}}.\end{aligned}$$

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Pivot

If $\boldsymbol{r}_B$ has a negative component, increasing the corresponding non-basic variable decreases the objective.

Choose an entering index $q \in N = [n] \setminus B$ such that $r_{B,q} < 0$, and define the direction d by

$$\begin{aligned} d_q &= 1, \\ d_{N\setminus\{q\}} &= 0, \\ d_B &= -A_B^{-1} A_q. \end{aligned}$$

Then $A(\check{x} + td) = b$ for every t, and

$$c^\top(\check{x} + td) = c^\top \check{x} + t\, r_{B,q}.$$

Thus the objective decreases as $t > 0$ increases: we move in this direction as far as feasibility allows.

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$$\begin{aligned} d_q &= 1, \\ d_{N\setminus\{q\}} &= 0, \\ d_B &= -A_B^{-1} A_q. \end{aligned}$$

Then $A(\check{x} + td) = b$ for every t, and

$$c^\top(\check{x} + td) = c^\top \check{x} + t\, r_{B,q}.$$

Thus the objective decreases as $t > 0$ increases: we move in this direction as far as feasibility allows.

Ratio test and basis update

Feasibility along the pivot direction requires

$$\check{x}_i + td_i \geq 0 \qquad \forall i \in B.$$

- If $d_i \geq 0$ for every $i \in B$, then the direction remains feasible for all $t \geq 0$ and the problem is unbounded below.
- Otherwise, the largest feasible step is given by the minimum-ratio test

$$t^\star = \min_{i \in B:\, d_i < 0} \frac{\check{x}_i}{-d_i}.$$

Choose a leaving index

$$p \in \arg \min_{i \in B:\, d_i < 0} \frac{\check{x}_i}{-d_i},$$

and set $\widetilde{B} = (B \cup \{q\}) \setminus \{p\}$.

Then $\widetilde{B}$ is a feasible basis. If $t^\star = 0$, the pivot is degenerate: the basis changes but the current point does not.

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Simplex method

Input: A linear program in standard form and an initial feasible basis B.
Output: An optimal basis, or a certificate of unboundedness.

while *true* **do**

- Compute $\boldsymbol{r}_B = (\boldsymbol{c}_N^\top - \boldsymbol{c}_B^\top A_B^{-1} A_N)^\top$;
- **if** $\boldsymbol{r}_B \geq 0$ **then**
  - **return** B;
- Choose $q \in N$ such that $r_{B,q} < 0$;
- Compute $d_q = 1$, $d_{N \setminus \{q\}} = 0$, $d_B = -A_B^{-1} A_q$;
- **if** $d_i \geq 0$ *for every* $i \in B$ **then**
  - **return** "The problem is unbounded";
- Choose $p \in \arg \min_{i \in B:\, d_i < 0} \dfrac{\check{x}_i}{-d_i}$;
- Set $B \leftarrow (B \cup \{q\}) \setminus \{p\}$;

Algorithm 1: Simplex method (conceptual form)

Finding an initial feasible basis: Phase I

After multiplying some equality constraints by -1 if necessary, assume $b \geq 0$.
Introduce artificial variables $z \in \mathbb{R}^m_+$ and solve

$$\begin{aligned} \underset{x,z}{\text{Min}} \quad & \mathbf{1}^\top z \\ \text{s.t.} \quad & Ax + z = b, \\ & x \geq 0, \ z \geq 0. \end{aligned}$$

- $x = 0$, $z = b$ gives an obvious initial feasible basis (the artificial variables).
- The Phase I optimal value is 0 if and only if the original LP is feasible.
- At value 0, artificial variables can be pivoted out to recover a feasible basis for the original problem.

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Simplex variants and remarks

- The entering index q can be chosen with various pivot rules (most negative reduced cost, estimated objective decrease, ...).
- If the minimum-ratio test has several minimizers, a pivot rule selects the leaving index. Degenerate pivots may lead to cycling; suitable rules guarantee finite termination.
- Basis systems are maintained efficiently using matrix factorizations and rank-one updates; A_B^{-1} is not recomputed from scratch.
- The simplex method has exponential worst-case examples, but is very efficient in practice. Smoothed analysis partly explains this behavior. Linear programming also admits polynomial-time algorithms, notably interior-point methods.

Contents

Dual of a linear program

- The dual of the standard-form linear program

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{Min}} \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

  is

$$\begin{aligned} \underset{\lambda\in\mathbb{R}^m}{\text{Max}} \quad & b^\top \lambda \\ \text{s.t.} \quad & A^\top \lambda \leq c. \end{aligned}$$

- The dual variable λ is the vector of Lagrange multipliers associated with $Ax = b$.
- The dual of the dual is equivalent to the primal.

Reduced costs as a dual certificate

Let B be a feasible basis and define

$$\lambda_B := (A_B^{-1})^\top c_B.$$

Then

$$A_B^\top \lambda_B = c_B, \qquad \boldsymbol{r}_B = c_N - A_N^\top \lambda_B.$$

Therefore

$$\boxed{\boldsymbol{r}_B \geq 0 \iff A^\top \lambda_B \leq c,}$$

so the reduced-cost stopping condition is exactly dual feasibility.
Moreover, for the basic feasible solution $\check{\boldsymbol{x}}$ associated with B,

$$b^\top \lambda_B = c_B^\top A_B^{-1} b = c^\top \check{\boldsymbol{x}}.$$

Hence primal and dual feasible points have the same value: by weak duality, they are both optimal.

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Strong duality

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Theorem (Strong duality)

If the primal or dual linear program is feasible, then their optimal values are equal (possibly as extended values).

If either the primal or dual linear program has an optimal solution, then both have an optimal solution and their values are equal.

- If the primal is unbounded below (value $-\infty$), then the dual is infeasible.
- If the dual is unbounded above (value $+\infty$), then the primal is infeasible.
- The only case with a duality gap is when both problems are infeasible: the primal value is $+\infty$ and the dual value is $-\infty$.

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What you should know

- What a linear program is, and that its feasible set is a polyhedron.
- If an LP is feasible and bounded below, an optimum exists and one can be found at a vertex.
- The principle of the simplex method and the role of reduced costs and pivots.
- Linear programming satisfies strong duality, except in the fully degenerate case where both primal and dual are infeasible.

What you should be able to do

- Convert a linear program between the basic forms introduced in the course.
- Identify half-spaces, polyhedra, faces and vertices in simple examples.
- Compute a basic solution, reduced costs, and perform one simplex pivot.
- Write the dual of a simple linear program.