Introduction to Optimization

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[1] Based in part on slides by F. Meunier.

Why should I bother to learn this stuff?

- Engineering is full of decisions: how much to produce, how to design a structure, how to allocate limited resources.
- Optimization provides a systematic way to turn these questions into a model that can be solved numerically.
- The same framework applies to very different fields: mechanics, energy, logistics, economics, machine learning, ...
- Constraints are often what make an engineering problem interesting: capacities, balances, geometry, physical limits, ...

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- What is optimization?
- Vocabulary
- Practice and tools in optimization

2 First theoretical elements
- Existence of an optimal solution
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4 Wrap-up

What is optimization?

To optimize $=$ to choose the best decision among feasible alternatives.

This is a natural objective in engineering. Three representative examples:

1. **Production planning.** How much of each product should we manufacture to maximize profit under limited resources?
2. **Traveling salesman.** In which order should we visit a set of cities to minimize total travel cost?
3. **Spring equilibrium.** Which displacement minimizes mechanical energy while satisfying boundary conditions?

These examples lead respectively to linear, discrete, and differentiable constrained optimization models.

A mathematical optimization problem

$$\begin{aligned} \mathrm{Min}_x \quad & f(x) \\ \text{s.t.} \quad & x \in X \end{aligned} \tag{P}$$

An optimization problem is specified by three objects:

- x: the decision variable — what we can choose;
- X: the feasible set — the constraints;
- f: the objective function (or criterion) — how we evaluate a decision.

We mostly use minimization; maximizing f is equivalent to minimizing $-f$.

Course roadmap

The course is organized around four main pillars:

1. **Constrained optimization**: Lagrange multipliers and KKT conditions;
2. **Convex optimization**: global optimality, duality, and first-order algorithms;
3. **Linear optimization**: polyhedra, simplex, and linear-programming duality;
4. **Mixed-integer linear optimization**: discrete modeling and branch-and-bound.

The objective is not only to solve small problems by hand, but to understand how to formulate, analyze, and solve optimization models with modern numerical tools.

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Local and global minima

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $X \subseteq \mathbb{R}^n$.

$x^\sharp$ is a local minimum of f on X if there exists a neighborhood V of $x^\sharp$ such that

$$f(x^\sharp) \leq f(x) \qquad \text{for all } x \in V \cap X.$$

$x^\sharp$ is a global minimum of f on X if

$$f(x^\sharp) \leq f(x) \qquad \text{for all } x \in X.$$

$$\text{global minimum} \quad \Longrightarrow \quad \text{local minimum.}$$

Domain and sublevel sets

In optimization it is convenient to allow an objective to take the value $+\infty$. We denote

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}.$$

Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$.

- The domain of f is

$$\operatorname{dom} f := \{x \in \mathbb{R}^n \mid f(x) < +\infty\}.$$

- The sublevel set at level $M \in \mathbb{R}$ is

$$\operatorname{lev}_M(f) := \{x \in \mathbb{R}^n \mid f(x) \leq M\}.$$

Assigning $+\infty$ outside a set is a useful way to encode constraints inside an objective.

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Optimal solution and optimal value

$$\boxed{\begin{array}{ll} \mathrm{Min}_{x} & f(x) \\ \text{s.t.} & x \in X \end{array}}$$

$x^\sharp$ is an optimal solution if it is a global minimum of f on X.

The optimal value is

$$p^\star := \inf\{f(x) \mid x \in X\}.$$

The optimal value need not be attained by an optimal solution.

By convention, $p^\star = +\infty$ when $X = \varnothing$.

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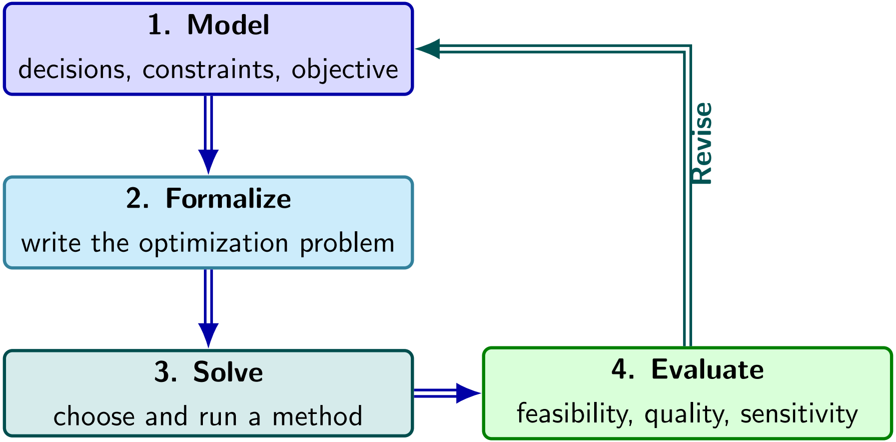
Optimization in practice: an iterative loop



An unsatisfactory solution may indicate a numerical issue or a modeling issue.

Optimization tools

- **Modeler** (e.g. AMPL, GAMS, Pyomo, JuMP, CVXPY): expresses variables, constraints, objectives, and indexed models in a readable form.

➥ First write the mathematical optimization problem clearly, then translate it into code.

- **Solver** (e.g. CPLEX, Gurobi, MOSEK, IPOPT): implements numerical algorithms for one or several classes of optimization problems.

➥ In general, use established solvers rather than reimplementing numerical algorithms from scratch.

- **Post-processing**: check residuals and status, interpret the solution, perform sensitivity analyses, and visualize results.
- **Programming language** (e.g. Julia, Python, C++): handles data and orchestrates the full workflow.

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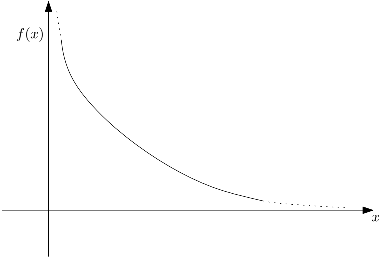
Optimal value versus optimal solution



A problem can have a finite optimal value but no optimal solution.

For example,

$$\begin{aligned} \mathrm{Min}_x \quad & \frac{1}{x} \\ \text{s.t.} \quad & x > 0. \end{aligned}$$

Its optimal value is 0, but no feasible x attains it.

Some topology recalls

- $C \subseteq \mathbb{R}^n$ is bounded if there exists $M > 0$ such that $\|x\| \leq M$ for all $x \in C$.
- $C \subseteq \mathbb{R}^n$ is closed if every convergent sequence in C has its limit in C.
- $C \subseteq \mathbb{R}^n$ is compact if it is closed and bounded.
- $f : \mathbb{R}^n \to \mathbb{R}$ is continuous if $x_k \to x$ implies $f(x_k) \to f(x)$.
- $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is coercive if

$$f(x) \to +\infty \qquad \text{when } \|x\| \to +\infty.$$

Theorem (Extreme value theorem)

A continuous real-valued function on a nonempty compact set attains its minimum and its maximum.

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Some existence results

Consider

$$\begin{aligned} \mathrm{Min}_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & x \in C. \end{aligned}$$

Theorem

If C is nonempty and compact and f is continuous, then the problem admits an optimal solution.

Theorem

If C is nonempty and closed and f is continuous and coercive, then the problem admits an optimal solution.

These are sufficient existence conditions, not necessary ones.

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Unconstrained optimization

♡

The first-order necessary condition is also known as Fermat's rule.

Theorem (FONC)

Let $x^\sharp$ be a local minimum of $f : \mathbb{R}^n \to \overline{\mathbb{R}}$. If f is differentiable at $x^\sharp$, then

$$\nabla f(x^\sharp) = \mathbf{0}.$$

A second-order sufficient condition is:

Theorem (SOSC)

Assume $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is twice differentiable on its domain. If $x^\sharp \in \operatorname{dom} f$, $\nabla f(x^\sharp) = \mathbf{0}$, and $\nabla^2 f(x^\sharp)$ is positive definite, then $x^\sharp$ is a local minimum.

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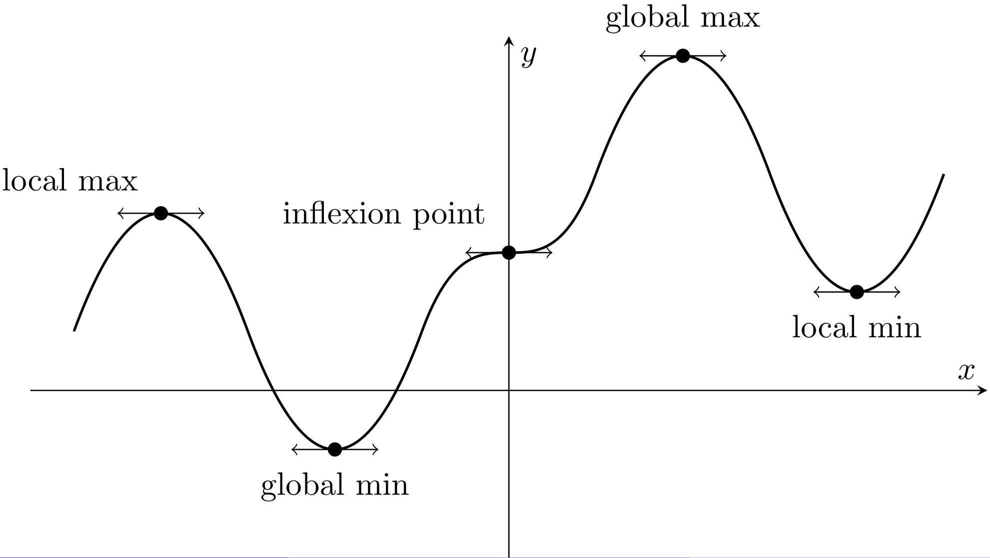
Critical points



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Equality constraints: Lagrange multipliers

♡

Consider

$$\begin{array}{lll} \mathrm{Min}_{x \in \mathbb{R}^n} & f(x) & \\ \text{s.t.} & g_i(x) = 0, & i \in [n_E], \end{array}$$

where f and the g_i are differentiable, and let

$$X := \{x \in \mathbb{R}^n \mid g_i(x) = 0, \ i \in [n_E]\}.$$

Theorem (Lagrange multipliers)

Let $x^\sharp$ be a local minimum of f on X. Under a technical regularity condition on the constraints at $x^\sharp$, there exists $\boldsymbol{\lambda} \in \mathbb{R}^{n_E}$ such that

$$\nabla f(x^\sharp) + \sum_{i=1}^{n_E} \lambda_i \nabla g_i(x^\sharp) = \mathbf{0}.$$

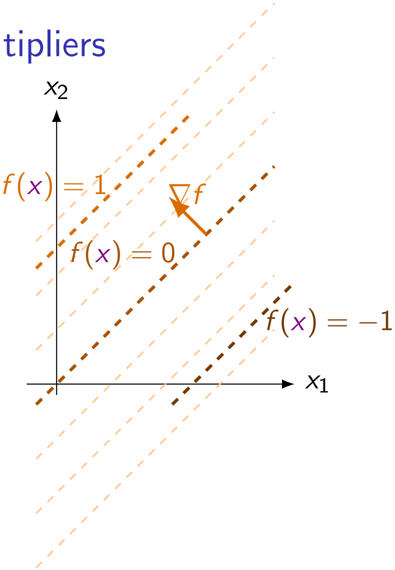
We will make the regularity condition precise later.

Geometric interpretation of Lagrange multipliers

$$f(x) = x_2 - x_1$$
$$g(x) = x_1^2 - x_2$$
$$X = \{x \in \mathbb{R}^2 : g(x) = 0\}$$

$$\nabla f(x) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Level lines of f are parallel, and ∇f is orthogonal to them.



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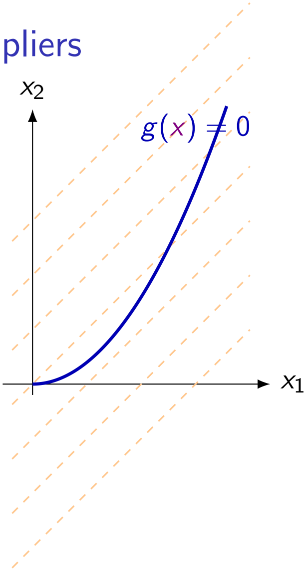
$$X = \{(x_1, x_2) : x_2 = x_1^2\}$$

The feasible set is the parabola $g(x) = 0$.



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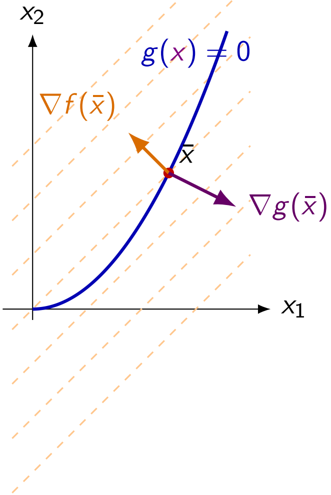
$$X = \{x \in \mathbb{R}^2 : g(x) = 0\}$$

$$\bar{x} = (1,1)$$

$$\nabla f(\bar{x}) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \nabla g(\bar{x}) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

At $\bar{x}$, the gradients are not collinear:
$\bar{x}$ is not a constrained optimum.



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$$f(x) = x_2 - x_1$$

$$g(x) = x_1^2 - x_2$$

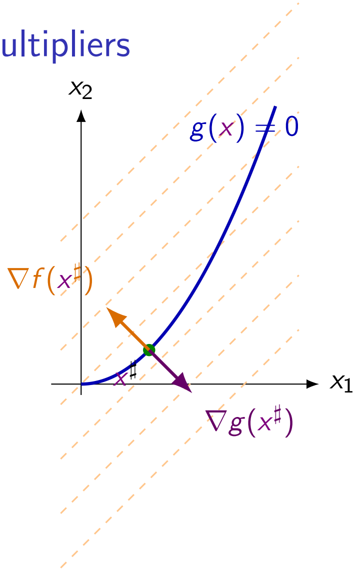
$$X = \{x \in \mathbb{R}^2 : g(x) = 0\}$$

$$x^\sharp = \left(\tfrac{1}{2}, \tfrac{1}{4}\right)$$

$$\nabla f(x^\sharp) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \nabla g(x^\sharp) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

At $x^\sharp$, the gradients are collinear:

$$\nabla f(x^\sharp) + \nabla g(x^\sharp) = \mathbf{0}.$$

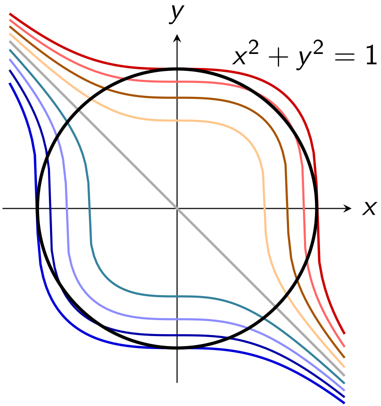


Lagrange multipliers: example

Consider

$$\begin{array}{ll} \text{Min} & x^3 + y^3 \\ \text{s.t.} & x^2 + y^2 = 1. \end{array}$$

The feasible set is the unit circle. The objective is represented on the right by its level lines.



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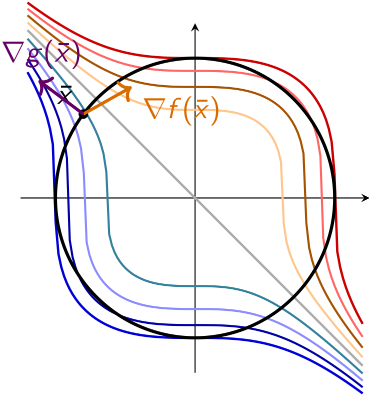

$$\nabla f(x,y) = \begin{pmatrix} 3x^2 \\ 3y^2 \end{pmatrix}, \qquad \nabla g(x,y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix}.$$

At an optimum, the two gradients must be collinear.

Lagrange multipliers: example

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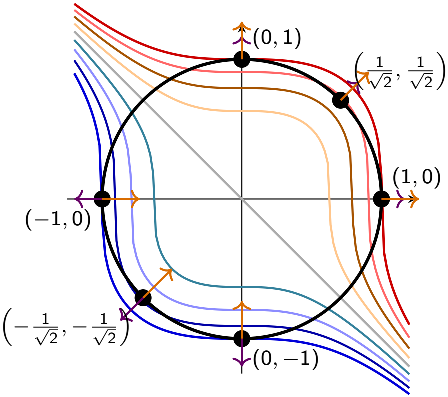

At any optimal solution $(x^\sharp, y^\sharp)$, there exists $\lambda \in \mathbb{R}$ such that

$$\begin{cases} 3x^{\sharp 2} + 2\lambda x^\sharp = 0, \\ 3y^{\sharp 2} + 2\lambda y^\sharp = 0, \\ x^{\sharp 2} + y^{\sharp 2} = 1. \end{cases}$$

Hence every optimum is among

$$(\pm 1, 0),\ (0, \pm 1),\ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),\ \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

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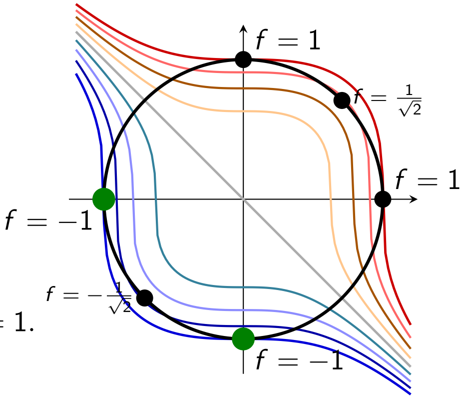

Comparing objective values gives

$$f(-1,0) = f(0,-1) = -1,$$

$$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}, \qquad f(1,0) = f(0,1) = 1.$$

Therefore the two global minimizers are

$$\boxed{(-1,0) \qquad (0,-1)}.$$

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Unconstrained first-order optimality

Let $x^\sharp$ be a local minimum of a differentiable function f.
For any direction $d \in \mathbb{R}^n$ and $t > 0$ small enough,

$$f(x^\sharp + td) = f(x^\sharp) + t\nabla f(x^\sharp) \cdot d + o(t).$$

Since $x^\sharp$ is a local minimum,

$$f(x^\sharp + td) - f(x^\sharp) \geq 0,$$

hence, dividing by t and letting $t \downarrow 0$,

$$\nabla f(x^\sharp) \cdot d \geq 0 \qquad \forall d \in \mathbb{R}^n.$$

Choose $d = -\nabla f(x^\sharp)$:

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What does local optimality imply under constraints? $\diamondsuit$

Let $x^\sharp$ be a local minimum of f on X. Consider

$$x^k \in X, \qquad x^k \to x^\sharp.$$

By differentiability, for k large enough,

$$f(x^k) = f(x^\sharp) + \nabla f(x^\sharp)^\top (x^k - x^\sharp) + o(\|x^k - x^\sharp\|) \geq f(x^\sharp)$$

Hence

$$\nabla f(x^\sharp)^\top (x^k - x^\sharp) + o(\|x^k - x^\sharp\|) \geq 0.$$

Set

$$t_k := \|x^k - x^\sharp\|, \qquad \boldsymbol{d}^k := \frac{x^k - x^\sharp}{t_k}.$$

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The directions allowed by the constraints

Since $\|\boldsymbol{d}^k\| = 1$, we can extract a subsequence such that

$$\boldsymbol{d}^k \to d.$$

Passing to the limit,

$$\nabla f(x^\sharp)^\top d \geq 0.$$

This motivates collecting all directions that can arise from feasible points approaching $x^\sharp$.

Definition (Tangent cone)

$$T_X(x^\sharp) := \left\{ d \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists\, t_k \searrow 0,\ \exists\, \boldsymbol{d}^k \to d, \\ x^\sharp + t_k \boldsymbol{d}^k \in X \end{array} \right\}.$$

Therefore, at a constrained local minimum,

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The directions allowed by the constraints

Since $\|\boldsymbol{d}^k\| = 1$, we can extract a subsequence such that

$$\boldsymbol{d}^k \to d.$$

Passing to the limit,

$$\nabla f(x^\sharp)^\top d \geq 0.$$

This motivates collecting all directions that can arise from feasible points approaching $x^\sharp$.

Definition (Tangent cone)

$$T_X(x^\sharp) := \left\{ d \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists\, t_k \searrow 0,\ \exists\, \boldsymbol{d}^k \to d, \\ x^\sharp + t_k \boldsymbol{d}^k \in X \end{array} \right\}.$$

Therefore, at a constrained local minimum,

$$\boxed{\nabla f(x^\sharp)^\top d \geq 0 \qquad \forall d \in T_X(x^\sharp).}$$

Linearizing the constraints

Suppose now

$$X = \{x \in \mathbb{R}^n \mid g_i(x) = 0, \; i \in [n_E]\}.$$

Take $d \in T_X(x^\sharp)$ and corresponding feasible points x^k.
Since $g_i(x^k) = g_i(x^\sharp) = 0$,

$$0 = g_i(x^k) - g_i(x^\sharp) = t_k \nabla g_i(x^\sharp) \cdot d + o(t_k).$$

Therefore every tangent direction satisfies

$$\nabla g_i(x^\sharp) \cdot d = 0 \qquad \forall i \in [n_E].$$

Define the linearized tangent cone

$$T_X^\ell(x^\sharp) := \left\{ d \in \mathbb{R}^n \mid \nabla g_i(x^\sharp) \cdot d = 0, \; \forall i \right\}.$$

Hence

$$\boxed{T_X(x^\sharp) \subseteq T_X^\ell(x^\sharp).}$$

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Constraint qualification and Lagrange multipliers

Linearization may introduce directions that are not truly feasible.

We say that the constraints are qualified at $x^\sharp$ if

$$\boxed{T_X(x^\sharp) = T_X^\ell(x^\sharp).}$$

Let G be the Jacobian of $g = (g_1, \dots, g_{n_E})$. Under qualification,

$$\nabla f(x^\sharp) \cdot d \geq 0 \qquad \forall d \in \ker G.$$

But $\ker G$ is a vector space: if $d \in \ker G$, then $-d \in \ker G$. Thus

$$\nabla f(x^\sharp) \cdot d = 0 \qquad \forall d \in \ker G.$$

Therefore

$$\nabla f(x^\sharp) \in (\ker G)^\perp = \mathrm{Im}(G^\top),$$

so there exists λ such that

$$\boxed{\nabla f(x^\sharp) + G^\top \lambda = 0.}$$

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Simple sufficient qualification conditions

How can we check qualification in practice?

Proposition

If all equality constraints g_i are affine, then they are qualified at every feasible point.

Corollary (LICQ)

Assume the g_i are continuously differentiable near x. If

$$\{\nabla g_i(x) : i \in [n_E]\}$$

is linearly independent, then the constraints are qualified at x.

These conditions are sufficient, not necessary.

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What you should know

- The basic vocabulary of an optimization problem: variables, objective, constraints, feasible set, local/global optimum, optimal solution/value.
- The main sufficient conditions ensuring existence of an optimal solution.
- The first- and second-order optimality conditions for unconstrained differentiable problems.
- The Lagrange multiplier condition for equality-constrained problems.
- The practical workflow: model $\rightarrow$ formalize $\rightarrow$ solve $\rightarrow$ evaluate/revise.

What you should be able to do

- Formulate a simple optimization problem from an engineering question.
- Check standard sufficient conditions for existence of an optimum.
- Use first- and second-order conditions to study a small unconstrained problem.
- Solve a small equality-constrained problem using Lagrange multipliers.