Decomposition-coordination method for the management of a chain of dams

J-C. Alais, P. Carpentier, V. Leclère

September 26, 2012

Model presentation

Parameters :

- the storage level x_t^i,
- the hydroturbine outflows u_t^i,
- the external inflows w_t^i,
- the selling prices p_t^i.

Objective function :

$$\mathbb{E}\left[\sum_{i=1}^{N}\sum_{t=0}^{T-1}\underbrace{-p_t\, u_t^i + \epsilon(u_t^i)^2}_{=L_t^i(x_t^i,u_t^i,w_t^i,z_t^i)} + K_i(x_T)\right]$$

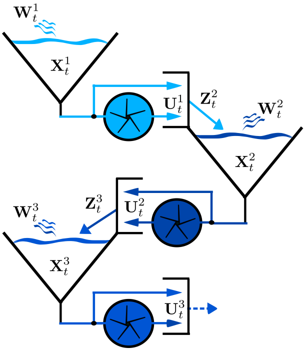


Figure: The river chain model

Formal problem presentation

Thus the stochastic optimization problem we are solving reads

$$\min_{(\mathbf{X},\mathbf{U},\mathbf{Z})} \mathbb{E}\Bigg(\sum_{i=1}^{N} \Big(\sum_{t=0}^{T-1} L_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i, \mathbf{W}_t^i, \mathbf{Z}_t^i) + K^i(x_T^i) \Big)\Bigg), \qquad (1a)$$

subject to:

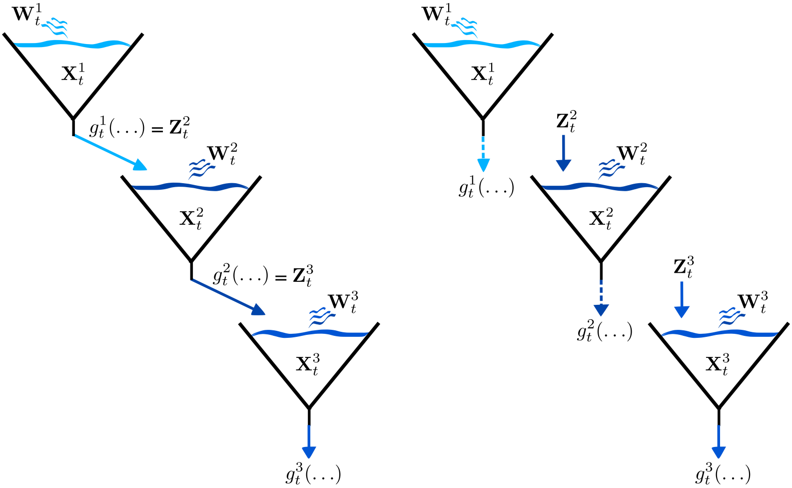
$$\forall i\ ,\ \forall t \qquad \mathbf{X}_{t+1}^i = f_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i, \mathbf{W}_t^i, \mathbf{Z}_t^i)\ , \qquad (1b)$$

$$\forall i\ ,\ \forall t \qquad \mathbf{Z}_t^{i+1} = g_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i, \mathbf{W}_t^i, \mathbf{Z}_t^i)\ , \qquad (1c)$$

as well as measurability constraints:

$$\forall i\ ,\ \forall t \qquad \mathbf{U}_t^i \preceq \mathcal{F}_t\ . \qquad (1d)$$

Decomposition Principle



(a) (b)

We aim at dualizing the coupling Constraint and at solving the Problem (1) by using the Uzawa algorithm: at iteration k, the associated multiplier is a fixed $\mathcal{F}_t$-measurable random variable $\left(\boldsymbol{\lambda}_t^{i+1}\right)^{(k)}$, and the term (under the expectation) induced by duality in the cost function is

$$\left(\boldsymbol{\lambda}_t^{i+1}\right)^{(k)} . \left(\mathbf{Z}_t^{i+1} - g_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i, \mathbf{W}_t^i, \mathbf{Z}_t^i)\right) .$$

It can be decomposed as

- $\left(\boldsymbol{\lambda}_t^{i+1}\right)^{(k)} . \mathbf{Z}_t^{i+1}$: term pertaining to dam $i+1$.
- $-\left(\boldsymbol{\lambda}_t^{i+1}\right)^{(k)} . g_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i, \mathbf{W}_t^i, \mathbf{Z}_t^i)$: term pertaining to dam i.

Consequently the algorithm is done as follow :

1. we fix multipliers $\left(\boldsymbol{\lambda}_t^{i}\right)^{(k)}$ for all i and t,
2. we have to solve N problems with only one dam,
3. we update the multiplier by a gradient step.

One dam problem

Consequently optimization problem associated to dam i at iteration k of the Uzawa algorithm is:

$$\min_{(\mathbf{X}^i,\mathbf{U}^i,\mathbf{Z}^i)} \mathbb{E}\Bigg(\sum_{t=0}^{T-1} \Big(L_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i, \mathbf{W}_t^i, \mathbf{Z}_t^i) + (\boldsymbol{\lambda}_t^i)^{(k)} . \mathbf{Z}_t^i - (\boldsymbol{\lambda}_t^{i+1})^{(k)} . g_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i, \mathbf{W}_t^i, \mathbf{Z}_t^i) \Big) + K^i(x_T^i) \Bigg), \tag{2a}$$

$$\mathbf{X}_{t+1}^i = f_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i, \mathbf{W}_t^i, \mathbf{Z}_t^i) , \ \forall t , \tag{2b}$$

$$\mathbf{U}_t^i \preceq \mathcal{F}_t \quad \text{and} \quad \mathbf{Z}_t^i \preceq \mathcal{F}_t , \ \forall t . \tag{2c}$$

This problem is a one dimensional dam problem and can be solved by **DP** or by any other method.

DADP principle

- $\big(\boldsymbol{\lambda}_t^i\big)_{t=0,\dots,T-1}^{(k)}$ prevents us to use **DP**.
- The idea of **DADP** is to replace $\big(\boldsymbol{\lambda}_t^i\big)^{(k)}$ by $\mathbb{E}\Big(\big(\boldsymbol{\lambda}_t^i\big)^{(k)} \mid \mathbf{Y}_t^i\Big)$,
- or equivalently replace the constraint $\mathbf{Z}_t^i = g_t^{i-1}\big(\mathbf{X}_t^{i-1}, \mathbf{U}_t^{i-1}, \mathbf{W}_t^{i-1}, \mathbf{Z}_t^{i-1}\big)$ by

$$\mathbb{E}\Big(\mathbf{Z}_t^i - g_t^{i-1}\big(\mathbf{X}_t^{i-1}, \mathbf{U}_t^{i-1}, \mathbf{W}_t^{i-1}, \mathbf{Z}_t^{i-1}\big) \;\Big|\; \mathbf{Y}_t^i\Big) . \tag{3}$$

In practice, the information variable $\mathbf{Y}_t^i$ is a short-memory process that will enter the state variables of the subproblems. Possible choices for $\mathbf{Y}_t^i$ are:

1. $\mathbf{Y}_t^i = \text{const}$: we deal with the constraint in expectation,
2. $\mathbf{Y}_t^i = \mathbf{W}_t^{i-1}$: we incorporate the noise $\mathbf{W}_t^{i-1}$ in Subproblem i,
3. $\mathbf{Y}_t^i = \tilde{f}_t^{i-1}\big(\mathbf{Y}_{t-1}^i, \mathbf{W}_t^{i-1}\big)$: we mimic the dynamics of $\mathbf{X}_t^{i-1}$.

DADP principle

- $\left(\boldsymbol{\lambda}_t^i\right)_{t=0,\dots,T-1}^{(k)}$ prevents us to use **DP**.
- The idea of **DADP** is to replace $\left(\boldsymbol{\lambda}_t^i\right)^{(k)}$ by $\mathbb{E}\left(\left(\boldsymbol{\lambda}_t^i\right)^{(k)} \mid \mathbf{Y}_t^i\right)$,
- or equivalently replace the constraint $\mathbf{Z}_t^i = g_t^{i-1}\left(\mathbf{X}_t^{i-1}, \mathbf{U}_t^{i-1}, \mathbf{W}_t^{i-1}, \mathbf{Z}_t^{i-1}\right)$ by

$$\mathbb{E}\left(\mathbf{Z}_t^i - g_t^{i-1}\left(\mathbf{X}_t^{i-1}, \mathbf{U}_t^{i-1}, \mathbf{W}_t^{i-1}, \mathbf{Z}_t^{i-1}\right) \;\middle|\; \mathbf{Y}_t^i\right) . \tag{3}$$

In practice, the information variable $\mathbf{Y}_t^i$ is a short-memory process that will enter the state variables of the subproblems. Possible choices for $\mathbf{Y}_t^i$ are:

1. $\mathbf{Y}_t^i \equiv \mathrm{const}$: we deal with the constraint in expectation,
2. $\mathbf{Y}_t^i = \mathbf{W}_t^{i-1}$: we incorporate the noise $\mathbf{W}_t^{i-1}$ in Subproblem i,
3. $\mathbf{Y}_t^i = \widetilde{f}_t^{i-1}\left(\mathbf{Y}_{t-1}^i, \mathbf{W}_t^{i-1}\right)$: we mimic the dynamics of $\mathbf{X}_t^{i-1}$.

Detailed Algorithm 1/4

We give here a formal presentation of the algorithm. First the initialization of the algorithm should be done as follow

- We fix some random particles (that is some trajectories of the noise) $(W_t^l)_{t\in[0,T]}$ that will be used throughout the algorithm.
- We initialize $\left(\boldsymbol{\lambda}_t^i\right)^{(0)}$ as deterministic well chosen constants (zero by default), and $\left(\varphi_t^i\right)^{(0)}$ as constant functions.

Then at the beginning of iteration k we should have defined

- A variable of information $\left(\mathbf{Y}_t^i\right)^{(k)}$ which should be an (uncontrolled process)

$$\mathbf{Y}_t^i = \tilde{f}_t^i\left(\mathbf{Y}_{t-1}^i, \xi_t^i\right) .$$

- A function $\left(\varphi_t^i\right)^{(k)}$ such that

$$\left(\varphi_t^i\right)^{(k)}(y) \approx \mathbb{E}\left(\left(\boldsymbol{\lambda}_t^i\right)^{(k)} \mid \left(\mathbf{Y}_t^i\right)^{(k)} = y\right)$$

Detailed Algorithm 1/4

We give here a formal presentation of the algorithm. First the initialization of the algorithm should be done as follow

- We fix some random particles (that is some trajectories of the noise) $(W_t^l)_{t\in[0,T]}$ that will be used throughout the algorithm.
- We initialize $\left(\boldsymbol{\lambda}_t^i\right)^{(0)}$ as deterministic well chosen constants (zero by default), and $\left(\varphi_t^i\right)^{(0)}$ as constant functions.

Then at the beginning of iteration k we should have defined

- A variable of information $\left(\mathbf{Y}_t^i\right)^{(k)}$ which should be an (uncontrolled process)

$$\mathbf{Y}_t^i = \widetilde{f}_t^i\left(\mathbf{Y}_{t-1}^i, \boldsymbol{\xi}_t^i\right) .$$

- A function $\left(\varphi_t^i\right)^{(k)}$ such that

$$\left(\varphi_t^i\right)^{(k)}(y) \approx \mathbb{E}\left(\left(\boldsymbol{\lambda}_t^i\right)^{(k)} \mid \left(\mathbf{Y}_t^i\right)^{(k)} = y\right)$$

Detailed Algorithm 2/4

For each i we solve

$$\min_{\mathbf{X}^i,\mathbf{U}^i,\mathbf{Z}^i} \quad \mathbb{E}\Big[\sum_{t=0}^{T} L_t^i(\mathbf{X}_t^i,\mathbf{U}_t^i,\mathbf{W}_t^i) + (\varphi_t^i)^{(k)}(\mathbf{Y}_t^i).\mathbf{Z}_t^i - (\varphi_t^{i+1})^{(k)}(\mathbf{Y}_t^{i+1}).g(\mathbf{X}_t^i,\mathbf{U}_t^i,\mathbf{W}_t^i,\mathbf{Z}_t^i)\Big]$$

$$\mathbf{X}_{t+1}^i = f_t^i(\mathbf{X}_t^i,\mathbf{U}_t^i,\mathbf{Z}_t^i,\mathbf{W}_t^i)$$

$$\mathbf{Y}_{t+1}^i = (\tilde{f}^i)_t^{(k)}(\mathbf{Y}_t^i,\xi_t^i)$$

$$\mathbf{Y}_{t+1}^{i+1} = (\tilde{f}^{i+1})_t^{(k)}(\mathbf{Y}_t^{i+1},\xi_t^{i+1})$$

$$\mathbf{U}_t^i \preceq \mathcal{F}_t$$

$$\mathbf{Z}_t^i \preceq \mathcal{F}_t$$

Detailed Algorithm 3/4

This gives us some optimal feedback laws

- $(\gamma_t^i)^{(k)}\Big(\mathbf{X}_t^i, \mathbf{Y}_t^i, \mathbf{Y}_t^{i+1}, \mathbf{W}_t^i, \xi_t^i, \xi_t^{i+1}\Big) \quad \rightsquigarrow \quad \mathbf{U}_t^i$
- $(\eta_t^i)^{(k)}\Big(\mathbf{X}_t^i, \mathbf{Y}_t^i, \mathbf{Y}_t^{i+1}, \mathbf{W}_t^i, \xi_t^i, \xi_t^{i+1}\Big) \quad \rightsquigarrow \quad \mathbf{Z}_t^i$

that are used with
$(\mathbf{X}_t^{i,l})^{(k)}$, $(\mathbf{U}_t^{i,l})^{(k)}$, $(\mathbf{Z}_t^{i,l})^{(k)}$, $(\mathbf{Y}_t^{i,l})^{(k)}$, $(\mathbf{Y}_t^{i+1,l})^{(k)}$, to compute

$$(\mathbf{U}_t^{i,l})^{(k)} = (\gamma_t^i)^{(k)}\Big((\mathbf{X}_t^i)^{(k)}, (\mathbf{Y}_t^i)^{(k)}, (\mathbf{Y}_t^{i+1})^{(k)}, \mathbf{W}_t^{i,l}, \xi_t^{i,l}, \xi_t^{i+1,l}\Big)$$

$$(\mathbf{Z}_t^{i,l})^{(k)} = (\eta_t^i)^{(k)}\Big((\mathbf{X}_t^i)^{(k)}, (\mathbf{Y}_t^i)^{(k)}, (\mathbf{Y}_t^{i+1})^{(k)}, \mathbf{W}_t^{i,l}, \xi_t^{i,l}, \xi_t^{i+1,l}\Big)$$

and

$$(\mathbf{X}_{t+1}^{i,l})^{(k)} = f_t^i\Big((\mathbf{X}_t^{i,l})^{(k)}, (\mathbf{U}_t^{i,l})^{(k)}, (\mathbf{Z}_t^{i,l})^{(k)}, \mathbf{W}_t^{i,l}\Big)$$

$$(\mathbf{Y}_{t+1}^{i,l})^{(k)} = (\tilde{f}^i)_t^{(k)}\Big((\mathbf{Y}_t^{i,l})^{(k)}, \xi_t^{i,l}\Big)$$

Detailed Algorithm 4/4

And finally we can

- Update of the prices trajectories:

$$\left(\boldsymbol{\lambda}_t^{i+1,l}\right)^{(k+1)} := \left(\boldsymbol{\lambda}_t^{i+1,l}\right)^{(k)} + \rho^{(k)}\left(\boldsymbol{\Delta}_t^{i,l}\right)^{(k)} ,$$

with $\left(\boldsymbol{\Delta}_t^{i,l}\right)^{(k)} :=$
$\left(\mathbf{Z}_t^{i+1,l}\right)^{(k)} - g_t^i\left(\left(\mathbf{X}_t^{i,l}\right)^{(k)}, \left(\mathbf{U}_t^{i,l}\right)^{(k)}, \mathbf{W}_t^{i,l}, \left(\mathbf{Z}_t^{i,l}\right)^{(k)}\right)$.

- Define a new information dynamics $\left(\tilde{f}^i\right)_t^{(k+1)}$.
- Simulate $\left(\mathbf{Y}_t^{i,l}\right)^{(k+1)}$.
- Make a regression of $\left(\boldsymbol{\lambda}_t^{i,l}\right)^{(k+1)}$ on $\left(\mathbf{Y}_t^{i,l}\right)^{(k+1)}$ to obtain

$$\left(\varphi_t^i\right)^{(k+1)}(y) \approx \mathbb{E}\left(\left(\boldsymbol{\lambda}_t^i\right)^{(k+1)} \;\middle|\; \left(\mathbf{Y}_t^i\right)^{(k+1)} = y\right) .$$

which terminate step k.

Back to admissibility

- Once the algorithm has converged we have some feedbacks laws that must verify the relaxed coupling constraint.
- Consequently we need an heuristic to obtain an admissible solution of the original problem.
- We suggest to approximate Bellman's value function for the global problem as the sum of the Bellman's value function of each subproblem where $\mathbf{Y}_t^i$ is replaced by $\mathbf{X}_t^i$.
- Consequently we obtain a global admissible strategy by doing a one time step optimization of the global problem.

Parameters and results

The characteristics of the study are:

- {min, max} bounds on $\mathbf{X}_t^i = \{0, 80\}\ hm^3, \forall (i,\ t)$;
- time steps number $T = 12$ (one step a month over a year);
- {min, max} bounds on $\mathbf{U}_t^i = \{0, 40\}\ hm^3 month^{-1}, \forall (i,\ t)$.

The stochastic universe is finite. The noise processes are white and uniformly distributed and the inflows at the three dams reservoirs are correlated. The simulation is based on 500 inflows scenarios.

We choose a constant information variable, thus we turn an almost sure constraint into a expected constraint.

We need about 3000 iterations to converges. The approximation that we make by estimating the multipliers as their expected values leads to a loss of about 1%. This is all the more promising that we use the simplest information variable.

Parameters and results

The characteristics of the study are:

- {min, max} bounds on $\mathbf{X}_t^i = \{0, 80\}\ hm^3, \forall(i, t)$;
- time steps number $T = 12$ (one step a month over a year);
- {min, max} bounds on $\mathbf{U}_t^i = \{0, 40\}\ hm^3 month^{-1}, \forall(i, t)$.

The stochastic universe is finite. The noise processes are white and uniformly distributed and the inflows at the three dams reservoirs are correlated. The simulation is based on 500 inflows scenarios.
We choose a constant information variable, thus we turn an almost sure constraint into a expected constraint.
We need about 3000 iterations to converges. The approximation that we make by estimating the multipliers as their expected values leads to a loss of about 1%. This is all the more promising that we use the simplest information variable.

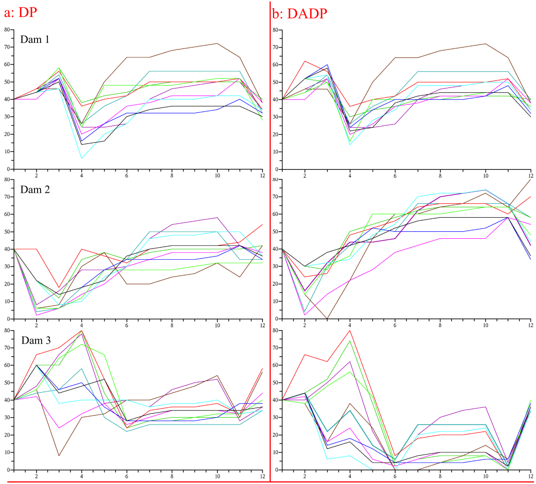


Figure: Six storage level trajectories

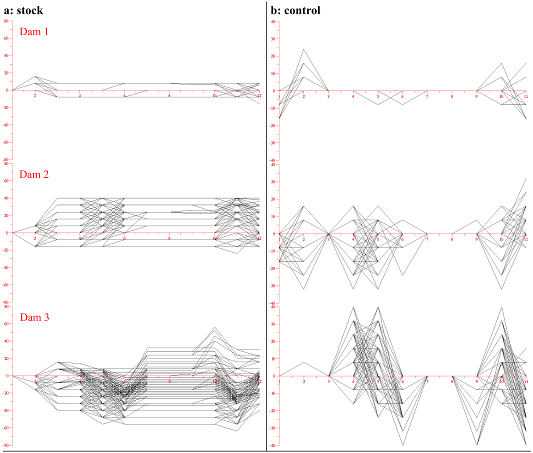


Figure: Differences in stock and controls on the 500 simulation scenarios ω

Conclusion

- Price decomposition in a stochastic setting is unpracticable.
- However we can solve a relaxed version of the original problem.
- In order to do that we have to choose an information variable, and replace the lagrange multiplier by its conditionnal expectation.
- Numerical results are promising, but there is still a lot of difficulties to use smart information variable.

Barty, K., Carpentier, P. and Girardeau, P. (2010)
Decomposition of large-scale stochastic optimal control problems.
RAIRO Operations Research, 2010, 44, 167-183.

Barty, K., Carpentier, P., Cohen, G. and Girardeau, P. (2010)
Price decomposition in large-scale stochastic optimal control.
arXiv, 2010, math.OC, 1012.2092.

The end

Thank you for your attention !