Uzawa Algorithm in $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$

V. Leclère (ENPC), P. Carpentier (ENSTA)

Conférence MODE, Rennes
March 27, 2014

What is this about ?

- We want to treat constraints in a stochastic optimization problem, by duality methods.
- Uzawa algorithm is a simple dual method: it is a gradient algorithm for the dual problem.
- Uzawa algorithm is naturally described in an Hilbert space, thus in L^2, but conditions of convergence in stochastic optimization fails: we cannot guarantee the existence of an optimal multiplier.
- Consequently, we extend the algorithm to the non-reflexive Banach $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ and gives a result of convergence.
- We also give conditions of existence of optimal multiplier.
- Finally we apply the algorithm to a multistage problem.

What is this about ?

- We want to treat constraints in a stochastic optimization problem, by duality methods.
- Uzawa algorithm is a simple dual method: it is a gradient algorithm for the dual problem.
- Uzawa algorithm is naturally described in an Hilbert space, thus in L^2, but conditions of convergence in stochastic optimization fails: we cannot guarantee the existence of an optimal multiplier.
- Consequently, we extend the algorithm to the non-reflexive Banach $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ and gives a result of convergence.
- We also give conditions of existence of optimal multiplier.
- Finally we apply the algorithm to a multistage problem.

What is this about ?

- We want to treat constraints in a stochastic optimization problem, by duality methods.
- Uzawa algorithm is a simple dual method: it is a gradient algorithm for the dual problem.
- Uzawa algorithm is naturally described in an Hilbert space, thus in L^2, but conditions of convergence in stochastic optimization fails: we cannot guarantee the existence of an optimal multiplier.
- Consequently, we extend the algorithm to the non-reflexive Banach $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ and gives a result of convergence.
- We also give conditions of existence of optimal multiplier.
- Finally we apply the algorithm to a multistage problem.

What is this about ?

- We want to treat constraints in a stochastic optimization problem, by duality methods.
- Uzawa algorithm is a simple dual method: it is a gradient algorithm for the dual problem.
- Uzawa algorithm is naturally described in an Hilbert space, thus in L^2, but conditions of convergence in stochastic optimization fails: we cannot guarantee the existence of an optimal multiplier.
- Consequently, we extend the algorithm to the non-reflexive Banach $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ and gives a result of convergence.
- We also give conditions of existence of optimal multiplier.
- Finally we apply the algorithm to a multistage problem.

Presentation Outline

1. Problem Statement and Hilbert Case
   - Problem Statement
   - Uzawa Algorithm in Hilbert Spaces
   - L^2 not Adapted for Almost Sure Constraint
2. Uzawa Algorithm in $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$
   - Differences Between $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ and an Hilbert space
   - Uzawa in $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$
   - Existence of L^1-multiplier
3. Application to a Multistage Problem
   - Multistage setup
   - Convergence Result and Remarks
4. Conclusion

Contents

1. Problem Statement and Hilbert Case
   - Problem Statement
   - Uzawa Algorithm in Hilbert Spaces
   - L^2 not Adapted for Almost Sure Constraint
2. Uzawa Algorithm in $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$
   - Differences Between $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ and an Hilbert space
   - Uzawa in $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$
   - Existence of L^1-multiplier
3. Application to a Multistage Problem
   - Multistage setup
   - Convergence Result and Remarks
4. Conclusion

Problem Statement

We consider the following (primal) problem:

$$(\mathcal{P}) \qquad \min_{u \in \mathcal{U}^{\mathrm{ad}}} \quad J(u) \ , \qquad s.t. \quad \Theta(u) \in -C \ .$$

Where $\mathcal{U}$ and $\mathcal{V}$ are two Hausdorff spaces, and

- $J : \mathcal{U} \to \bar{\mathbb{R}}$ is an objective function ,
- $\Theta : \mathcal{U} \to \mathcal{V}$ is a constraint function (to be dualized),
- $C \subset \mathcal{V}$ is a cone of constraints,
- $\mathcal{U}^{\mathrm{ad}} \subset \mathcal{U}$ is a constraint set (not to be dualized).

Dual Problem

The primal problem can be written

$$(\mathcal{P}) \qquad \min_{u \in \mathcal{U}^{\mathrm{ad}}} \quad \max_{\lambda \in C^\star} \qquad J(u) + \left\langle \lambda \,, \Theta(u) \right\rangle_{\mathcal{V}^\star, \mathcal{V}} \,,$$

where $C^\star \subset \mathcal{V}^\star$ is given by

$$C^\star = \left\{ \lambda \in \mathcal{V}^\star \mid \forall x \in C, \quad \left\langle \lambda \,, x \right\rangle_{\mathcal{V}^\star, \mathcal{V}} \geq 0 \right\} \,.$$

The dual problem of Problem $(\mathcal{P})$ reads

$$(\mathcal{D}) \qquad \max_{\lambda \in C^\star} \quad \min_{u \in \mathcal{U}^{\mathrm{ad}}} \qquad J(u) + \left\langle \lambda \,, \Theta(u) \right\rangle_{\mathcal{V}^\star, \mathcal{V}} \,.$$

Equivalence of $(\mathcal{P})$ and $(\mathcal{D})$, Saddle-Point and Multiplier.

We introduce the Lagrangian associated to Problem $(\mathcal{P})$,

$$L(u,\lambda) := J(u) + \langle \lambda \,, \Theta(u) \rangle_{\mathcal{V}^\star,\mathcal{V}} \,.$$

Proposition

The primal problem $(\mathcal{P})$ and the dual problem $(\mathcal{D})$ are equivalent (same value and same set of solutions), i.e,

$$\min_{u\in\mathcal{U}^{\mathrm{ad}}} \max_{\lambda\in C^\star} L(u,\lambda) = \max_{\lambda\in C^\star} \min_{u\in\mathcal{U}^{\mathrm{ad}}} L(u,\lambda) \,,$$

iff the Lagrangian L admits a saddle point on $\mathcal{U}^{\mathrm{ad}} \times C^\star$, or equivalently if the constraint $\Theta(u) \in -C$ is qualified.

Contents

1. Problem Statement and Hilbert Case
   - Problem Statement
   - Uzawa Algorithm in Hilbert Spaces
   - L^2 not Adapted for Almost Sure Constraint
2. Uzawa Algorithm in $L^\infty(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^n)$
   - Differences Between $L^\infty(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^n)$ and an Hilbert space
   - Uzawa in $L^\infty(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^n)$
   - Existence of L^1-multiplier
3. Application to a Multistage Problem
   - Multistage setup
   - Convergence Result and Remarks
4. Conclusion

Gradient of the Dual

Assume that $\mathcal{U} = \mathcal{U}^\star$, and $\mathcal{V} = \mathcal{V}^\star$ are Hilbert spaces.
Recall the dual problem $(\mathcal{D})$ as

$$\max_{\lambda \in C^\star} \underbrace{\min_{u \in \mathcal{U}^{\mathrm{ad}}} \Big\{ J(u) + \big\langle \lambda \,, \Theta(u) \big\rangle_{\mathcal{V}^\star, \mathcal{V}} \Big\}}_{:= \varphi(\lambda)} .$$

Under some regularity conditions, if $u^\sharp(\lambda)$ is a minimizer of the above problem, then

$$\Theta\big(u^\sharp(\lambda)\big) = \nabla \varphi(\lambda) .$$

$$\begin{cases} u^{(k)} & \in \arg\min_{u \in \mathcal{U}^{\mathrm{ad}}} \Big\{ J(u) + \big\langle \lambda^{(k)} \,, \Theta(u) \big\rangle_{\mathcal{V}^\star, \mathcal{V}} \Big\} \\ \lambda^{(k+1)} & = \mathrm{proj}_{C^\star} \Big(\lambda^{(k)} + \rho \, \Theta\big(u^{(k)}\big) \Big) \end{cases}$$

Uzawa Algorithm

Data: Initial multiplier $\lambda^{(0)} \in \mathcal{V}$, step $\rho > 0$;
Result: Optimal solution $u^\sharp$ and multiplier $\lambda^\sharp$;
repeat

$$u^{(k)} \in \arg\min_{u \in \mathcal{U}^{\mathrm{ad}}} \Big\{ J(u) + \langle \lambda^{(k)}, \Theta(u) \rangle \Big\} ,$$

$$\lambda^{(k+1)} = \mathrm{proj}_{C^\star} \Big(\lambda^{(k)} + \rho\, \Theta\big(u^{(k)}\big) \Big) .$$

until $\Theta(u^{(k)}) \in -C$;

Convergence of Uzawa Algorithm in Hilbert Spaces

proposition

Assume that,

1. the function $J : \mathcal{U} \to \bar{\mathbb{R}}$ is strongly convex of modulus a, and Gâteaux-differentiable;
2. the function $\Theta : \mathcal{U} \to \mathcal{V}$ is C-convex, and κ-Lipschitz;
3. $\mathcal{U}^{\mathrm{ad}} \neq \emptyset$ is a closed convex subset of the Hilbert space $\mathcal{U}$;
4. C is a non empty, closed convex cone of the Hilbert space $\mathcal{V}$;
5. the Lagrangian L admits a saddle-point $(u^\sharp, \lambda^\sharp)$ on $\mathcal{U}^{\mathrm{ad}} \times C^\star$;
6. the step size is small enough $(0 < \rho < 2a/\kappa^2)$.

Then, the Uzawa algorithm is well defined and, the sequence $\{u^{(k)}\}_{k\in\mathbb{N}}$ converges toward $u^\sharp$ in norm.

Convergence of Uzawa Algorithm in Hilbert Spaces

proposition

Assume that,

1. the function $J : \mathcal{U} \to \bar{\mathbb{R}}$ is strongly convex of modulus a, and Gâteaux-differentiable;
2. the function $\Theta : \mathcal{U} \to \mathcal{V}$ is C-convex, and κ-Lipschitz;
3. $\mathcal{U}^{\mathrm{ad}} \neq \emptyset$ is a closed convex subset of the Hilbert space $\mathcal{U}$;
4. C is a non empty, closed convex cone of the Hilbert space $\mathcal{V}$;
5. the Lagrangian L admits a saddle-point $(u^\sharp, \lambda^\sharp)$ on $\mathcal{U}^{\mathrm{ad}} \times C^\star$;
6. the step size is small enough ($0 < \rho < 2a/\kappa^2$).

Then, the Uzawa algorithm is well defined and, the sequence $\{u^{(k)}\}_{k \in \mathbb{N}}$ converges toward $u^\sharp$ in norm.

Contents

1. Problem Statement and Hilbert Case
   - Problem Statement
   - Uzawa Algorithm in Hilbert Spaces
   - L^2 not Adapted for Almost Sure Constraint
2. Uzawa Algorithm in $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$
   - Differences Between $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ and an Hilbert space
   - Uzawa in $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$
   - Existence of L^1-multiplier
3. Application to a Multistage Problem
   - Multistage setup
   - Convergence Result and Remarks
4. Conclusion

Stochastic Optimization Setting

In a stochastic optimization setting the most natural Hilbert space is $\mathrm{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. A natural optimization problem is thus

$$\min_{\mathbf{U} \in \mathcal{U}^{\mathrm{ad}} \subset \mathrm{L}^2} \quad \overbrace{\mathbb{E}\Big[j(\mathbf{U})\Big]}^{:=J(\mathbf{U})} = \int_\Omega j\big(\mathbf{U}(\omega), \omega\big) \, d\mathbb{P}(\omega) \; ,$$

$$s.t. \quad \Theta\big(\mathbf{U}\big) \in -C$$

where $j : \mathbb{R}^n \times \Omega \to \bar{\mathbb{R}}$ is a convex normal integrand (for example a Carathéodory integrand, that is continuous in u for almost all ω, and measurable in ω for all u).

Sufficient Condition of Qualification

Proposition

Under the following assumption

$$0 \in \mathrm{ri}\Big(\Theta\big(\mathcal{U}^{\mathrm{ad}} \cap \mathrm{dom}(J)\big) + C\Big) ,$$

The primal problem admits an optimal solution and constraint $\Theta\big(\mathbf{U}\big) \in -C$ is qualified.

Proposition

If the σ-algebra $\mathcal{F}$ is not finite, then for any set $U^{\mathrm{ad}} \subsetneq \mathbb{R}^n$, that is not a linear space, the set

$$\mathcal{U}^{\mathrm{ad}} = \Big\{\mathbf{U} \in \mathrm{L}^p\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big) \mid \mathbf{U} \in U^{\mathrm{ad}} \quad \mathbb{P} - a.s.\Big\} ,$$

has an empty (relative) interior in L^p, for $p < +\infty$.

Contents

1. Problem Statement and Hilbert Case
   - Problem Statement
   - Uzawa Algorithm in Hilbert Spaces
   - L^2 not Adapted for Almost Sure Constraint
2. Uzawa Algorithm in $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$
   - Differences Between $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ and an Hilbert space
   - Uzawa in $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$
   - Existence of L^1-multiplier
3. Application to a Multistage Problem
   - Multistage setup
   - Convergence Result and Remarks
4. Conclusion

L^∞ setting

From now on we consider that

$$
\begin{aligned}
\mathcal{U} &= \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n\big) \ , \\
\mathcal{V} &= \mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m\big) \ , \\
C &= \{0\}.
\end{aligned}
$$

Where the σ-algebra is not finite (modulo $\mathbb{P}$). Hence, $\mathcal{U}$ and $\mathcal{V}$ are non-reflexive, non-separable, Banach spaces.
If the σ-algebra is finite modulo $\mathbb{P}$, $\mathcal{U}$ and $\mathcal{V}$ are finite dimensional spaces, and the usual result applies.

Perks of an Hilbert Space

Fact

In an Hilbert space $\mathcal{H}$ we know that

i) the weak and weak* topologies are identical,
ii) the space $\mathcal{H}$ and its topological dual can be identified.

Point i) allows to formulate existence of minimizer results:

- weakly closed bounded $\Longrightarrow$ weakly compact;
- for a convex set : weakly closed $\Longleftrightarrow$ closed;
- for a convex function: weakly l.s.c $\Longleftrightarrow$ l.s.c.

Hence, a strongly-convex, lower semicontinuous function J admits an infimum.

Point ii) allows to write gradient-like algorithm: at any iteration k, we have a point $u^{(k)} \in \mathcal{H}$, and the gradient $g^{(k)} = \nabla f\big(u^{(k)}\big) \in \mathcal{H}$. Hence, linear combination of $\lambda^{(k)}$ and $g^{(k)}$ make sense.

Difficulties Appearing in a Banach Space

- In a reflexive Banach space E, *i*) still holds true, and thus the existence of a minimizer remains easy to show. However *ii*) does not hold anymore. Indeed g now belongs to the topological dual of E. Thus a combination of $u^{(k)} \in E$ and $g^{(k)} \in E^\star$ does not have any sense.
- In a non-reflexive Banach space E, neither *i*) nor *ii*) holds true.
- However if E is the topological dual of a Banach space, then a weakly* closed bounded subset of E is weak* compact. Thus, weak* lower semicontinuity and coercivity of a function J gives the existence of minimizers of J.

Specificities of $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$

- $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ is the topological dual of the Banach space $\mathrm{L}^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$. Hence, if J is weak* l.s.c and coercive, then J admits a minimizer.
- L^∞ can be identified with a subset of its topological dual $\left(\mathrm{L}^\infty\right)^\star$. Thus, the update step

$$\boldsymbol{\lambda}^{(k+1)} = \boldsymbol{\lambda}^{(k)} + \rho\, \Theta\big(\mathbf{U}^{(k)}\big) ,$$

make sense: it is a linear combination of elements of $\left(\mathrm{L}^\infty\right)^\star$.
- Moreover, if $\lambda^{(0)}$ is chosen in L^∞, then the sequence $\{\boldsymbol{\lambda}^{(k)}\}_{k\in\mathbb{N}}$ remains in L^∞.

Contents

1. Problem Statement and Hilbert Case
   - Problem Statement
   - Uzawa Algorithm in Hilbert Spaces
   - L^2 not Adapted for Almost Sure Constraint
2. Uzawa Algorithm in $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$
   - Differences Between $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ and an Hilbert space
   - Uzawa in $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$
   - Existence of L^1-multiplier
3. Application to a Multistage Problem
   - Multistage setup
   - Convergence Result and Remarks
4. Conclusion

Uzawa Algorithm

Data: Initial multiplier $\boldsymbol{\lambda}^{(0)} \in \mathrm{L}^\infty$, step $\rho > 0$;
Result: Optimal solution $\mathbf{U}^\sharp$ and multiplier $\boldsymbol{\lambda}^\sharp$;
repeat

$$\mathbf{U}^{(k)} \in \operatorname*{arg\,min}_{\mathbf{U} \in \mathcal{U}^{\mathrm{ad}}} \Big\{ J(\mathbf{U}) + \langle \boldsymbol{\lambda}^{(k)}, \Theta(\mathbf{U}) \rangle \Big\} ,$$

$$\boldsymbol{\lambda}^{(k+1)} = \boldsymbol{\lambda}^{(k)} + \rho\, \Theta\big(\mathbf{U}^{(k)}\big) .$$

until $\Theta(\mathbf{U}^{(k)}) = 0$;

Remark: numerically, other update rules (e.g. quasi-Newton) can be used, convergence being proven when we find a multiplier $\boldsymbol{\lambda}^{(k)}$ such that $\Theta\big(\mathbf{U}^{(k)}\big) = 0$.

Existence of Solution

Theorem

Assume that:

1. the constraint set $\mathcal{U}^{\mathrm{ad}}$ is weakly* closed;
2. $\Theta : \mathcal{U} \to \mathcal{V}$ is affine, weakly* continuous;
3. the objective function $J : \mathcal{U} \to \bar{\mathbb{R}}$ is weak* lower semicontinuous and coercive on $\mathcal{U}^{\mathrm{ad}}$;
4. there exists an admissible control.

Then the primal problem admits at least one solution.
Moreover for any $\boldsymbol{\lambda} \in \mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)$

$$\operatorname*{arg\,min}_{\mathbf{U} \in \mathcal{U}^{\mathrm{ad}}} \Big\{ J(\mathbf{U}) + \langle \boldsymbol{\lambda} \,, \Theta(\mathbf{U}) \rangle \Big\} \neq \emptyset \,.$$

Convergence Result

Theorem

Assume that:

1. $J : \mathcal{U} \to \bar{\mathbb{R}}$ is a proper, weak* lower semicontinuous, Gâteaux-differentiable, a-convex function;
2. $\Theta : \mathcal{U} \to \mathcal{V}$ is affine, weak* continuous and κ-Lipschitz;
3. there exists an admissible control;
4. $\mathcal{U}^{\mathrm{ad}}$ is weak* closed convex;
5. there is an optimal L^1-multiplier to the constraint $\Theta(\mathbf{U}) = 0$;
6. the step ρ is such that $0 < \rho < \frac{2a}{\kappa}$.

Then, Uzawa algorithm is well defined and there exists a subsequence $\left(\mathbf{U}^{(n_k)}\right)_{k\in\mathbb{N}}$ converging in L^∞ toward the optimal solution $\mathbf{U}^\sharp$ of the primal problem.

Convergence Result

Theorem

Assume that:

1. $J : \mathcal{U} \to \bar{\mathbb{R}}$ is a proper, weak* lower semicontinuous, Gâteaux-differentiable, a-convex function;
2. $\Theta : \mathcal{U} \to \mathcal{V}$ is affine, weak* continuous and κ-Lipschitz;
3. there exists an admissible control;
4. $\mathcal{U}^{\mathrm{ad}}$ is weak* closed convex;
5. there is an optimal L^1-multiplier to the constraint $\Theta(\mathbf{U}) = 0$;
6. the step ρ is such that $0 < \rho < \frac{2a}{\kappa}$.

Then, Uzawa algorithm is well defined and there exists a subsequence $\left(\mathbf{U}^{(n_k)}\right)_{k \in \mathbb{N}}$ converging in L^∞ toward the optimal solution $\mathbf{U}^\sharp$ of the primal problem.

Contents

1. Problem Statement and Hilbert Case
   - Problem Statement
   - Uzawa Algorithm in Hilbert Spaces
   - L^2 not Adapted for Almost Sure Constraint
2. Uzawa Algorithm in $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$
   - Differences Between $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ and an Hilbert space
   - Uzawa in $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$
   - Existence of L^1-multiplier
3. Application to a Multistage Problem
   - Multistage setup
   - Convergence Result and Remarks
4. Conclusion

Some Topologies on L^∞

- The topology $\tau_{\|\ \|}$ is the norm topology of L^∞.
- The weak topology $\sigma\left(\mathrm{L}^\infty, (\mathrm{L}^\infty)^\star\right)$ is the coarsest topology such that all norm-continuous linear form on L^∞ remains continuous.
- The weak* topology $\sigma\left(\mathrm{L}^\infty, \mathrm{L}^1\right)$ is the coarsest topology such that all the L^1-linear form are continuous.
- The Mackey-topology $\tau\left(\mathrm{L}^\infty, \mathrm{L}^1\right)$ is the finest topology such that the only continuous linear form are the L^1-linear form.

We have

$$\sigma(\mathrm{L}^\infty, (\mathrm{L}^\infty)^\star) \subset \tau(\mathrm{L}^\infty, \mathrm{L}^1) \subset \sigma(\mathrm{L}^\infty, \mathrm{L}^1) \subset \tau_{\|\ \|} .$$

- Coarser topology $\Longrightarrow$ more compact.
- Finer topology $\Longrightarrow$ more continuous real valued function.

Some Topologies on L^∞

- The topology $\tau_{\|\,\|}$ is the norm topology of L^∞.
- The weak topology $\sigma\Big(\mathrm{L}^\infty, (\mathrm{L}^\infty)^\star\Big)$ is the coarsest topology such that all norm-continuous linear form on L^∞ remains continuous.
- The weak* topology $\sigma\Big(\mathrm{L}^\infty, \mathrm{L}^1\Big)$ is the coarsest topology such that all the L^1-linear form are continuous.
- The Mackey-topology $\tau\Big(\mathrm{L}^\infty, \mathrm{L}^1\Big)$ is the finest topology such that the only continuous linear form are the L^1-linear form.

We have

$$\sigma\big(\mathrm{L}^\infty, (\mathrm{L}^\infty)^\star\big) \subset \tau\big(\mathrm{L}^\infty, \mathrm{L}^1\big) \subset \sigma\big(\mathrm{L}^\infty, \mathrm{L}^1\big) \subset \tau_{\|\|} \; .$$

- Coarser topology $\Longrightarrow$ more compact.
- Finer topology $\Longrightarrow$ more continuous real valued function.

A Theoretical Condition

Proposition

Assume that:

- $j : \mathbb{R}^d \times \Omega \to \bar{\mathbb{R}}$ is a convex normal integrand, such that
$$\exists\, \varepsilon > 0, \quad \exists\, \mathbf{U}_0 \in \mathcal{U}^{ad}, \quad \forall u \in \mathbb{R}^d,$$
$$\|u\|_{\mathbb{R}^d} \leq \varepsilon \quad \Longrightarrow \quad j(\mathbf{U}_0 + u, \cdot) < +\infty \quad \mathbb{P}-\text{a.s.}$$
- $J = \mathbb{E}\big[j(\cdot)\big]$ is $\tau\big(\mathrm{L}^\infty, \mathrm{L}^1\big)$-(upper-semi)continuous at some point $\mathbf{U}_0 \in \mathcal{U}^{ad} \cap \mathrm{dom}(J)$;
- $\mathcal{U}^{\mathrm{ad}}$ is a weak* closed linear subspace of $\mathrm{L}^\infty\big(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\big)$;

Then, the constraint $\Theta\big(\mathbf{U}\big) = 0$ admit a multiplier in L^1.

Remark : J is weak* l.s.c.

A Practical Condition

Proposition

Assume that j is a convex integrand and that and that J is finite everywhere on $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$. Then, J is $\tau(\mathrm{L}^\infty, \mathrm{L}^1)$-continuous.

Proposition

Consider a convex normal integrand $j : \mathbb{R}^n \times \Omega \to \bar{\mathbb{R}}$. Consider a set $U^{\mathrm{ad}} \subsetneq \mathbb{R}^m$ and define the set of random variable

$$\mathcal{U}^{\text{a.s.}} := \left\{ \mathbf{U} \in \mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \;\middle|\; \mathbf{U} \in U^{\mathrm{ad}} \quad \mathbb{P}-\text{a.s.} \right\}.$$

Then,

$$\tilde{J} : \mathbf{U} \mapsto J(\mathbf{U}) + \chi_{\mathbf{U} \in \mathcal{U}^{\text{a.s.}}},$$

is not Mackey continuous on its domain.

A Practical Condition

Proposition

Assume that j is a convex integrand and that and that J is finite everywhere on $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$. Then, J is $\tau(\mathrm{L}^\infty, \mathrm{L}^1)$-continuous.

Proposition

Consider a convex normal integrand $j : \mathbb{R}^n \times \Omega \to \bar{\mathbb{R}}$, Consider a set $U^{\mathrm{ad}} \subsetneq \mathbb{R}^m$ and define the set of random variable

$$\mathcal{U}^{\text{a.s.}} := \left\{ \mathbf{U} \in \mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \quad \middle| \quad \mathbf{U} \in U^{\mathrm{ad}} \quad \mathbb{P} - \text{a.s.} \right\} .$$

Then,

$$\widetilde{J} : \mathbf{U} \mapsto J(\mathbf{U}) + \chi_{\mathbf{U} \in \mathcal{U}^{\text{a.s.}}} ,$$

is not Mackey continuous on its domain.

Other Conditions with Relatively Complete Recourse Assumptions

- This Mackey-continuity assumption forbid the use of almost sure bounds.
- In order to deal with almost sure bounds, we can turn towards the work of R.T.Rockafellar and R.J-B.Wets. In a first series of 4 papers (stochastic convex programming) they detailed the duality on a two stage problem; which was extended to multistage problems in 3 other papers (with a specific focus on non-anticipativity constraints).
- These papers require:
  - a strict feasability assumption,
  - a relatively complete recourse assumption.

Contents

1. Problem Statement and Hilbert Case
   - Problem Statement
   - Uzawa Algorithm in Hilbert Spaces
   - L^2 not Adapted for Almost Sure Constraint
2. Uzawa Algorithm in $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$
   - Differences Between $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ and an Hilbert space
   - Uzawa in $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$
   - Existence of L^1-multiplier
3. Application to a Multistage Problem
   - Multistage setup
   - Convergence Result and Remarks
4. Conclusion

Problem Statement

$$
\begin{aligned}
\min_{\mathbf{X},\mathbf{D}} \quad & \mathbb{E}\Big[\sum_{t=0}^{T-1} L_t(\mathbf{X}_t, \mathbf{D}_t, \mathbf{W}_t) + K(\mathbf{X}_T)\Big] & \\
s.t. \quad & \mathbf{X}_0 = x_0 & \\
& \mathbf{X}_{t+1} = f_t(\mathbf{X}_t, \mathbf{D}_t, \mathbf{W}_t), & \text{dynamic} \\
& \mathbf{D}_t \preceq \sigma(\mathbf{W}_0, \dots, \mathbf{W}_t), & \text{non-anticipativity} \\
& \mathbf{D}_t \in \mathcal{D}_t^{\mathrm{ad}}, \qquad \mathbb{P} - \text{a.s.} & \text{bound constraint} \\
& \mathbf{X}_t \in \mathcal{X}_t^{\mathrm{ad}}, \qquad \mathbb{P} - \text{a.s.} & \text{bound constraint} \\
& \theta_t(\mathbf{X}_t, \mathbf{D}_t) = \mathbf{B}_t \qquad \mathbb{P} - \text{a.s.} & \text{affine constraint}
\end{aligned}
$$

Uzawa algorithm

Data: Initial multiplier process $\boldsymbol{\lambda}^{(0)} \in \mathrm{L}^\infty$, step $\rho > 0$;
Result: Optimal solution $\mathbf{D}^\sharp$ and multiplier process $\lambda^\sharp$;
repeat

$$\left(\mathbf{D}^{(k)}, \mathbf{X}^{(k)}\right) \in \quad \operatorname*{arg\,min}_{\mathbf{D},\mathbf{X}} \quad \Big\{ \mathbb{E}\Big[\sum_{t=0}^{T-1} L_t\big(\mathbf{X}_t, \mathbf{D}_t, \mathbf{W}_t\big) + \boldsymbol{\lambda}_t^{(k)} \cdot \theta_t\big(\mathbf{X}_t, \mathbf{D}_t\big) \Big] \Big\}$$

$$\boldsymbol{\lambda}_t^{(k+1)} = \quad \boldsymbol{\lambda}_t^{(k)} + \rho_t\Big(\theta_t\big(\mathbf{X}_t^{(k)}, \mathbf{D}^{(k)}\big) - \mathbf{B}_t\Big) \ .$$

where $(\mathbf{D}, \mathbf{X})$ satisfies all constraint except the dualized one.

until $\forall t \in [\![0, T]\!], \quad \theta_t\big(\mathbf{X}_t^{(k+1)}, \mathbf{D}^{(k+1)}\big) = \mathbf{B}_t$;

Contents

1. Problem Statement and Hilbert Case
   - Problem Statement
   - Uzawa Algorithm in Hilbert Spaces
   - L^2 not Adapted for Almost Sure Constraint
2. Uzawa Algorithm in $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$
   - Differences Between $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ and an Hilbert space
   - Uzawa in $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$
   - Existence of L^1-multiplier
3. Application to a Multistage Problem
   - Multistage setup
   - Convergence Result and Remarks
4. Conclusion

Convergence Result

Proposition

Assume that,

1. the cost functions L_t are Gâteaux-differentiable (in (x, u)), strongly-convex (in (x, u)) functions and continuous in w;
2. the constraint functions $\theta_t : \mathbb{R}^{n_x+n_d} \to \mathbb{R}^{n_c}$ are affine;
3. the evolution functions $f_t : \mathbb{R}^{n_x+n_d+n_w} \to \mathbb{R}^{n_x}$ are affine;
4. the constraint sets $\mathcal{X}_t^{\mathrm{ad}}$ and $\mathcal{U}_t^{\mathrm{ad}}$ are weak* closed, convex;
5. there exist a process $(\mathbf{X}, \mathbf{D})$ satisfying all constraints;
6. there exist an optimal multiplier process in L^1 to the almost sure affine constraint.

Then Uzawa algorithm is well defined, and there exists a subsequence $\left(\mathbf{D}^{(n_k)}\right)_{k\in\mathbb{N}}$ converging in L^∞ toward the optimal control of the multistage problem.

Remarks

- If there is no bound constraint, then there exist a L^1-multiplier.
- A multiplier $\boldsymbol{\lambda} = \{\boldsymbol{\lambda}_0, \dots, \boldsymbol{\lambda}_T\}$ is a stochastic process that can be chosen adapted with respect to $\mathfrak{F} = \{\mathcal{F}_0, \dots, \mathcal{F}_T\}$ where $\mathcal{F}_t := \sigma\big(\mathbf{W}_0, \dots, \mathbf{W}_t\big)$.
- However, if we want to use this algorithm as the master programm of a decomposition algorithm (by price) we have to solve, for a given adapted process $\boldsymbol{\lambda}^{(k)}$

$$\min_{\mathbf{D},\mathbf{X}} \quad \left\{ \mathbb{E}\Big[\sum_{t=0}^{T-1} L_t\big(\mathbf{X}_t, \mathbf{D}_t, \mathbf{W}_t\big) + \boldsymbol{\lambda}_t^{(k)} \cdot \theta_t\big(\mathbf{X}_t, \mathbf{D}_t\big) \Big] \right\} ,$$

  where $\big(\mathbf{D}, \mathbf{X}\big)$ satisfies all constraint except the dualized one.
- If we approximate the multiplier process $\boldsymbol{\lambda}$ by $\mathbb{E}[\boldsymbol{\lambda}_t \mid \mathbf{Y}_t]$, where $\mathbf{Y}_t$ is a Markov chain, then we can solve this minimization problem by DP (with the state $\big(\mathbf{X}_t, \mathbf{Y}_t\big)$.

In a nutshell

- Uzawa algorithm is a gradient algorithm for the dual problem, that naturally take place in Hilbert space, like L^2.
- Convergence result of Uzawa algorithm require the existence of an optimal multiplier of the dualized constraint.
- Sufficient conditions of existence of an optimal multiplier in L^2 are not adapted to almost sure constraint. L^∞ is better suited to this purpose.
- Consequently we have seen that Uzawa algorithm make sense in L^∞ and given a result of convergence (of a subsequence) that require a L^1 multiplier...
- and we have given conditions of existence of a L^1 multiplier.

The next steps

- Finally we have applied this algorithm to a multistage problem, and given conditions of convergence.
- However, there is two difficulties:
  - solving the minimization problem for a given $\boldsymbol{\lambda}^{(k)}$ is difficult;
  - the space of stochastic process in which we apply the gradient algorithm is very large.
- Hence, we propose to search the multiplier $\boldsymbol{\lambda}^{(k)}$ in a smaller space: $\boldsymbol{\lambda}_t$ is assumed to be measurable with respect to an information process $\mathbf{Y}_t$.
- Thus this algorithm can be used as the master problem of a (spatial) decomposition method in stochastic optimization.
- This is the Dual Approximate Dynamique Programming (DADP) algorithm. More ar SPO on 15th of April by P.Carpentier.

The end

Thank you for your attention !