A new look at the Stochastic Dual Dynamic Programming algorithm

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Introduction

We are interested in multistage stochastic optimization problems of the form

$$\begin{aligned} \min_{\pi} \quad & \mathbb{E}\Big[\sum_{t=0}^{T-1} L_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{\xi}_t) + K(\boldsymbol{X}_T)\Big] \\ s.t. \quad & \boldsymbol{X}_{t+1} = f_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{\xi}_t), \qquad \boldsymbol{X}_0 = x_0 \\ & \boldsymbol{U}_t = \pi_t(\boldsymbol{X}_t, \boldsymbol{\xi}_t) \in U_t(x, \boldsymbol{\xi}_t) \end{aligned}$$

where

- $\boldsymbol{x}_t$ is the state of the system,
- $\boldsymbol{u}_t$ is the control applied at time t,
- $\boldsymbol{\xi}_t$ is the noise happening between time t and $t+1$, assumed to be time-independent,
- π is the policy.

Time-decomposition

$$\min_{\pi} \quad \mathbb{E}\Big[L_0(x_0, \boldsymbol{U}_0, \xi_0) + \mathbb{E}\Big[\sum_{t=1}^{T-1} L_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \xi_t) + K(\boldsymbol{X}_T) \Big]\Big]$$

$$s.t. \quad \boldsymbol{X}_0 = x_0$$

$$\boldsymbol{X}_1 = f_t(\boldsymbol{X}_0, \boldsymbol{U}_0, \xi_0),$$

$$\boldsymbol{U}_0 = \pi_0(x_0, \xi_0) \in U_0(x_0, \xi_0)$$

$$\boldsymbol{X}_1 = f_t(\boldsymbol{X}_0, \boldsymbol{U}_0, \xi_0)$$

$$\boldsymbol{X}_{t+1} = f_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \xi_t)$$

$$\boldsymbol{U}_t = \pi_t(\boldsymbol{X}_t, \xi_t) \in U_t(\boldsymbol{X}_t, \xi_t)$$

Stochastic Dynamic Programming

By the white noise assumption, this problem can be solved by Dynamic Programming, where the Bellman functions satisfy

$$\begin{cases} V_T(x) &= K(x) \\ \hat{V}_t(x,\xi) &= \min\limits_{u_t \in U_t(x,\xi)} L_t(x,u_t,\xi) + V_{t+1} \circ \underbrace{f_t(x,u_t,\xi)}_{"x_{t+1}"} \\ V_t(x) &= \mathbb{E}\Big[\hat{V}_t(x,\boldsymbol{\xi}_t)\Big] \end{cases}$$

Indeed, π is an optimal policy if

$$\pi_t(x,\xi) \in \mathop{\arg\min}_{u_t \in U_t(x,\xi)} \Big\{ \underbrace{L_t(x,u_t,\xi)}_{\text{current cost}} + \underbrace{V_{t+1} \circ f_t(x,u_t,\xi)}_{\text{future costs}} \Big\}$$

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Bellman operator

For any time t, and any function $R : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ we define

$$\hat{\mathcal{T}}_t(R)(x,\xi) := \min_{u_t \in \mathbb{U}} L_t(x, u_t, \xi) + R \circ f_t(x, u_t, \xi)$$

and

$$\mathcal{T}_t(R)(x) := \mathbb{E}\Big[\hat{\mathcal{T}}_t(R)(x, \boldsymbol{\xi})\Big].$$

Incidentally, R induce a policy $\pi_t^R(x,\xi)$ given by a minimizer of the above problem, and an optimal policy is given by π^V.
Finally the Bellman equation simply reads

$$\begin{cases} V_T = K \\ V_t = \mathcal{T}_t(V_{t+1}) \end{cases}$$

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Properties of the Bellman operator

- Monotonicity:

$$V \leq \overline{V} \quad \Rightarrow \mathcal{T}_t\big(V\big) \leq \mathcal{T}_t\big(\overline{V}\big)$$

- Convexity: if L_t is jointly convex in (x, u), V is convex, and f_t is affine then

$$x \mapsto \mathcal{T}_t\big(V\big)(x) \quad \text{is convex}$$

- Polyhedrality: for any polyhedral function V, if L_t is also polyhedral, and f_t affine, then

$$x \mapsto \mathcal{T}_t\big(V\big)(x) \quad \text{is polyhedral}$$

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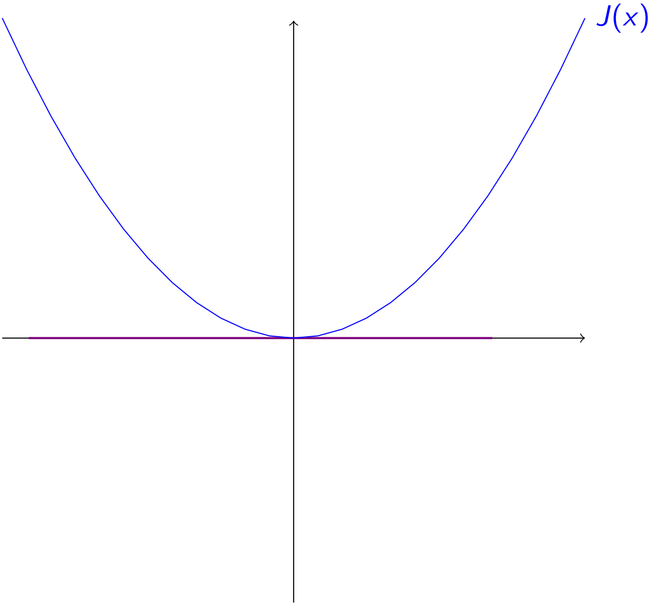
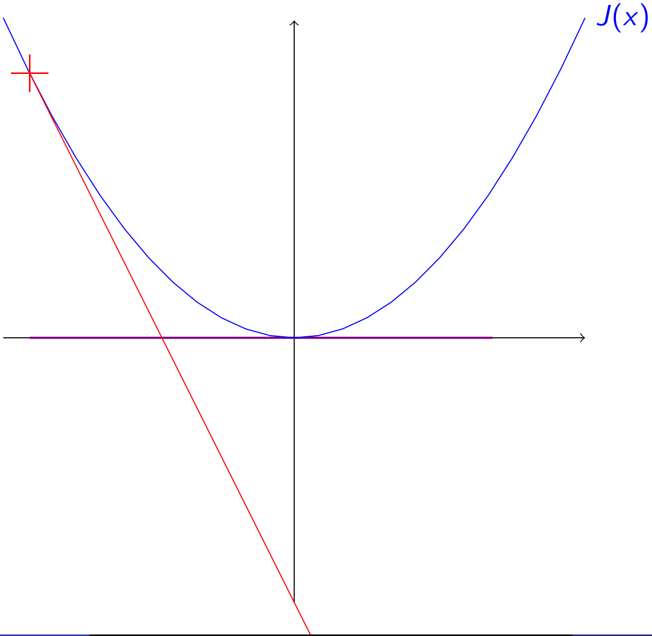

$J(x)$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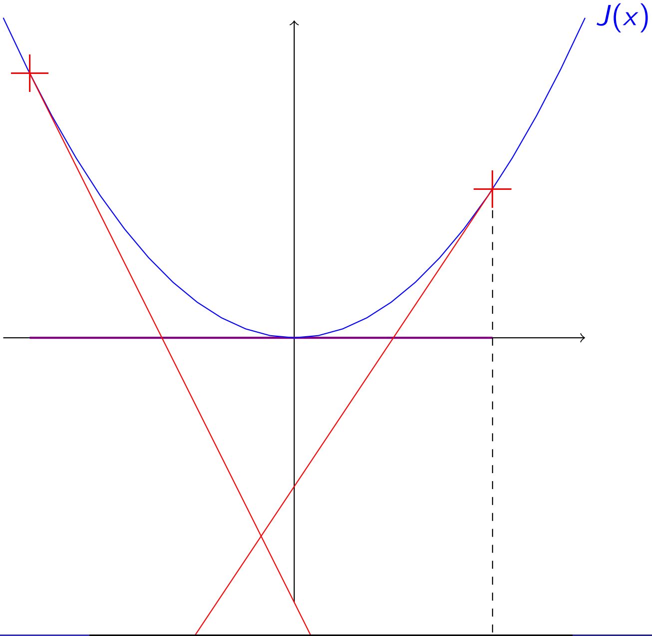
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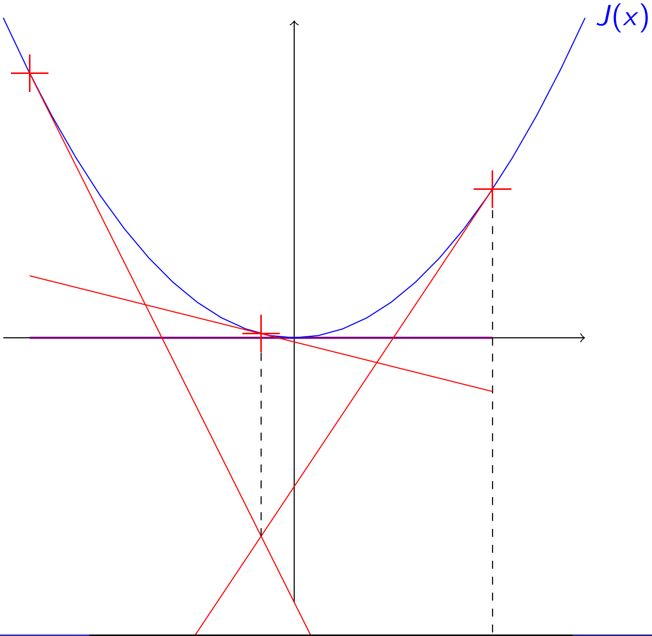
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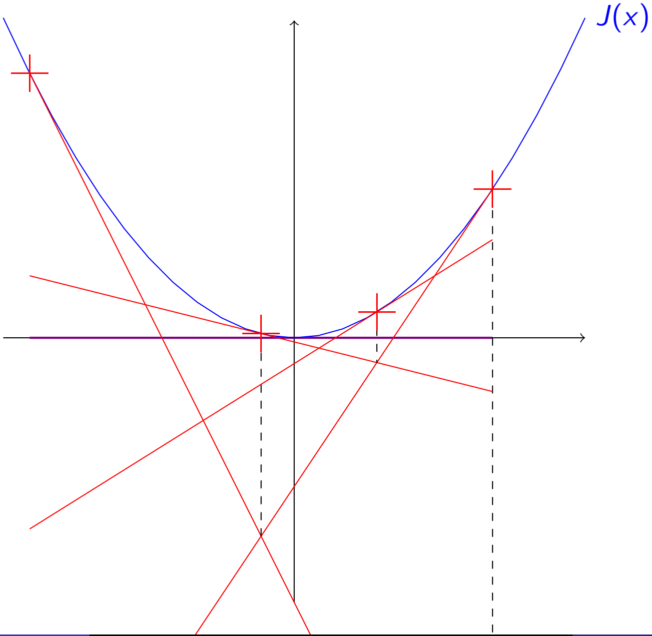
J(x)

Duality property

- Consider $J : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ jointly convex, and define

$$\varphi(x) = \min_{u \in \mathbb{U}} J(x, u)$$

- Then we can obtain a subgradient $\lambda \in \partial\varphi(x_0)$ as the dual multiplier of

$$\begin{aligned} \min_{x,u} \quad & J(x, u), \\ s.t. \quad & x_0 - x = 0 \qquad\qquad [\lambda] \end{aligned}$$

  (This is the marginal interpretation of the multiplier)
- In particular, we have that

$$\varphi(\cdot) \geq \varphi(x_0) + \langle \lambda, \cdot - x_0 \rangle$$

Computing cuts (1/2)

Suppose that we have $\underline{V}_{t+1}^{(k+1)} \leq V_{t+1}$

$$\begin{aligned} \hat{\beta}_t^{(k+1)} &= \min_{x,u} \quad L_t(x, u) + \underline{V}_{t+1}^{(k+1)} \circ f_t(x, u) \\ & s.t \quad x = x_t^{(k)} \qquad [\hat{\lambda}_t^{(k+1)}] \end{aligned}$$

This can also be written as

$$\begin{aligned} \hat{\beta}_t^{(k+1)} &= \hat{\mathcal{T}}_t\left(\underline{V}_{t+1}^{(k+1)}\right)(x) \\ \hat{\lambda}_t^{(k+1)} &\in \partial_x \hat{\mathcal{T}}_t\left(\underline{V}_{t+1}^{(k+1)}\right)(x) \end{aligned}$$

Thus, $\hat{\mathcal{C}}_t^{(k+1)} : x \mapsto \hat{\beta}_t^{(k+1)} + \left\langle \hat{\lambda}_t^{(k+1)}, x - x_t^{(k)} \right\rangle$ satisfy

$$\hat{\mathcal{C}}_t^{(k+1)}(x) \leq \hat{\mathcal{T}}_t\left(\underline{V}_{t+1}^{(k+1)}\right)(x) \leq \hat{\mathcal{T}}_t\left(V_{t+1}\right)(x) = \hat{V}_t(x)$$

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Computing cuts (1/2)

Suppose that we have $\underline{V}_{t+1}^{(k+1)} \leq V_{t+1}$

$$\hat{\beta}_t^{(k+1)}(\xi) = \min_{x,u} \quad L_t(x,u,\xi) + \underline{V}_{t+1}^{(k+1)} \circ f_t(x,u,\xi)$$
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Thus, $\hat{\mathcal{C}}_t^{(k+1),\xi} : x \mapsto \hat{\beta}_t^{(k+1)}(\xi) + \left\langle \hat{\lambda}_t^{(k+1)}(\xi), x - x_t^{(k)} \right\rangle$ satisfy, for all ξ,

$$\hat{\mathcal{C}}_t^{(k+1),\xi}(x) \leq \hat{\mathcal{T}}_t\left(\underline{V}_{t+1}^{(k+1)}\right)(x,\xi) \leq \hat{\mathcal{T}}_t\left(V_{t+1}\right)(x,\xi) = \hat{V}_t(x,\xi)$$

Computing cuts (2/2)

Thus,

$$\hat{\beta}_t^{(k+1)}(\xi) + \left\langle \hat{\lambda}_t^{(k+1)}(\xi), x - x_t^{(k)} \right\rangle \leq \hat{V}_t(x, \xi)$$

for each realization ξ of $\boldsymbol{\xi}_t$.
Replacing ξ by $\boldsymbol{\xi}_t$ and taking the expectation yields

$$\mathbb{E}\left[\hat{\beta}_t^{(k+1)}(\boldsymbol{\xi}_t)\right] + \mathbb{E}\left[\left\langle \hat{\lambda}_t^{(k+1)}(\boldsymbol{\xi}_t), x - x_t^{(k)} \right\rangle\right] \leq \mathbb{E}\left[\hat{V}_t(x, \boldsymbol{\xi}_t)\right] = V_t(x)$$

and finally we have the cut

$$\beta_t^{(k+1)} + \left\langle \lambda_t^{(k+1)}, \cdot - x_t^{(k)} \right\rangle \leq V_t$$

where

$$\begin{cases} \beta_t^{(k+1)} & := & \mathbb{E}\left[\hat{\beta}_t^{(k+1)}(\boldsymbol{\xi}_t)\right] = \mathcal{T}_t\left(\underline{V}_{t+1}^{(k)}\right)(x) \\ \lambda_t^{(k+1)} & := & \mathbb{E}\left[\hat{\lambda}_t^{(k+1)}(\boldsymbol{\xi}_t)\right] \in \partial \mathcal{T}_t\left(\underline{V}_{t+1}^{(k)}\right)(x) \end{cases}$$

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Contents

SDDP algorithm

Under linear dynamics, and convex costs, the SDDP algorithm iteratively constructs polyhedral outer approximations of V_t.

More precisely, at iteration k

- We have polyhedral functions $\underline{V}_t^k(\cdot) = \max_{\kappa \leq k} \left\{ \left\langle \lambda_t^\kappa , \cdot \right\rangle + \beta_t^\kappa \right\}$, such that $\underline{V}_t^k \leq V_t$.
- **Forward pass**: We simulate the dynamical system, along one scenario, according to $\pi^{\underline{V}^k}$, yielding a trajectory $\{\underline{x}_t^k\}_{t \in [\![0,T]\!]}$.
- **Backward pass**: We compute cuts

$$\mathcal{C}_t^k : x \mapsto \left\langle \lambda_t^{k+1} , x \right\rangle + \beta_t^{k+1} \leq V_t$$

along this trajectory, and update our outer approximations.

SDDP strengths

- SDDP is a widely used algorithm in the energy community, with multiple applications in
  - mid and long term water storage management problem,
  - long-term investment problems,
  - ...
- Recent works have presented extensions of the algorithm to
  - deal with some non-convexity,
  - treat risk-averse or distributionally robust problems,
  - incorporate integer variables.
- Multiple numerical improvements have been proposed
  - cut selection
  - regularization
  - multi-cut or ε-resolution

SDDP weaknesses

There are still some gaps in our knowledge of this approach:

- there is no convergence speed guaranteed,
- regularization methods are not mature yet,
- there is no good stopping test.

SDDP Stopping test

- Exact lower bound of the problem : $\underline{V}_0^k(x_0)$.
- Upper-bound estimated by Monte-Carlo simulation yielding costly statistical stopping tests (Pereira Pinto (1991) or Shapiro (2011))
- Alternative statistical tests have been proposed (see Homem de Mello et al (2011))
- Exact upper-bound computation has been proposed by Philpott et al (2013) but without any proof of convergence, leading to possibly not converging stopping tests.

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Linear Bellman Operator

An operator $\mathcal{B} : F(\mathbb{R}^{n_x}) \to F(\mathbb{R}^{n_x})$ is said to be a *linear Bellman operator* (LBO) if it is defined as follows

$$\mathcal{B}(R) : x \mapsto \inf_{(\boldsymbol{u},\boldsymbol{y})} \mathbb{E}\Big[\boldsymbol{c}^\top \boldsymbol{u} + R(\boldsymbol{y})\Big]$$

$$s.t. \quad Tx + \mathcal{W}_u(\boldsymbol{u}) + \mathcal{W}_y(\boldsymbol{y}) \leq \boldsymbol{h}$$

where $\mathcal{W}_u : \mathcal{L}^0(\mathbb{R}^{n_u}) \to \mathcal{L}^0(\mathbb{R}^{n_c})$ and $\mathcal{W}_y : \mathcal{L}^0(\mathbb{R}^{n_x}) \to \mathcal{L}^0(\mathbb{R}^{n_c})$ are two linear operators. We denote $S(R)(x)$ the set of $\boldsymbol{y}$ that are part of optimal solutions to the above problem.
We also define $\mathcal{G}(x)$

$$\mathcal{G}(x) := \big\{(\boldsymbol{u},\boldsymbol{y}) \mid Tx + \mathcal{W}_u(\boldsymbol{u}) + \mathcal{W}_y(\boldsymbol{y}) \leq \boldsymbol{h}\big\}$$

Examples

- Linear point-wise operator:

$$\begin{array}{rccc} \mathcal{W} & : & \mathcal{L}^0(\mathbb{R}^{n_x}) & \to & \mathcal{L}^0(\mathbb{R}^{n_c}) \\ & & (\omega \mapsto \boldsymbol{y}(\omega)) & \mapsto & (\omega \mapsto A\boldsymbol{y}(\omega)) \end{array}$$

  Such an operator allows to encode almost sure constraints.

- Linear expected operator:

$$\begin{array}{rccc} \mathcal{W} & : & \mathcal{L}^0(\mathbb{R}^{n_x}) & \to & \mathcal{L}^0(\mathbb{R}^{n_c}) \\ & & (\omega \mapsto \boldsymbol{y}(\omega)) & \mapsto & (\omega \mapsto A\,\mathbb{E}(\boldsymbol{y})) \end{array}$$

  Such an operator allows to encode constraints in expectation.

Relatively Complete Recourse and cuts

Definition (Relatively Complete Recourse)

We say that the pair $(\mathcal{B}, R)$ satisfy a *relatively complete recourse* (RCR) assumption if for all $x \in \mathrm{dom}(\mathcal{G})$ there exists admissible controls $(\boldsymbol{u}, \boldsymbol{y}) \in \mathcal{G}(x)$ such that $\boldsymbol{y} \in \mathrm{dom}(R)$.

Cut

If R is proper and polyhedral, with RCR assumption, then $\mathcal{B}(R)$ is a proper polyhedral function.

Furthermore, computing $\mathcal{B}(R)(x)$ consists of solving a linear problem which also generates a supporting hyperplane of $\mathcal{B}(R)$, that is, a pair $(\lambda, \beta) \in \mathbb{R}^{n_x} \times \mathbb{R}$ such that

$$\begin{cases} \langle \lambda \,, \cdot \rangle + \beta \leq \mathcal{B}(R)(\cdot) \\ \langle \lambda \,, x \rangle + \beta = \mathcal{B}(R)(x) \end{cases}$$

Contents

Setting

Consider a *compatible* sequence of LBO $\{\mathcal{B}_t\}_{t \in [\![0, T-1]\!]}$, that is, such that all admissible controls of $\mathcal{B}_t$ lead to admissible states of $\mathcal{B}_{t+1}$.

Consider a sequence of functions such that

$$\begin{cases} R_T = K \\ R_t = \mathcal{B}_t(R_{t+1}) \quad \forall t \in [\![0, T-1]\!] \end{cases}$$

Then, the abstract SDDP algorithm generates a sequence of lower polyhedral approximations of R_t. In a forward pass it simulates a trajectory of states, along which the approximation is refined in the backward pass.

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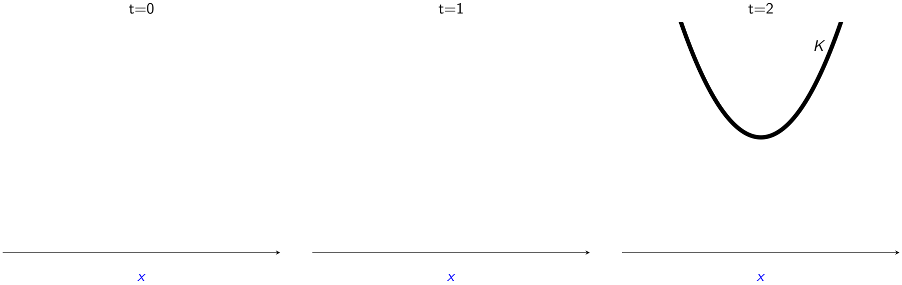
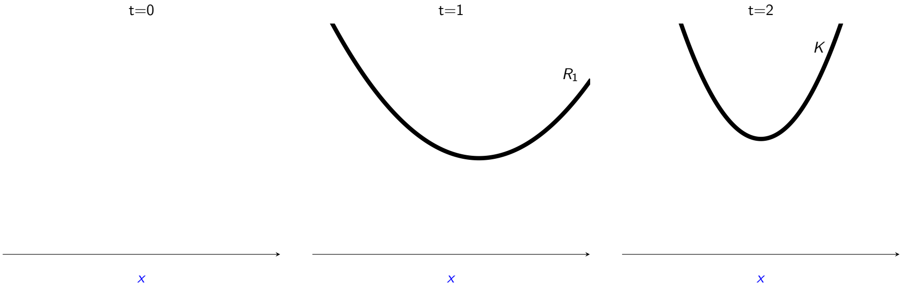
Abstract SDDP



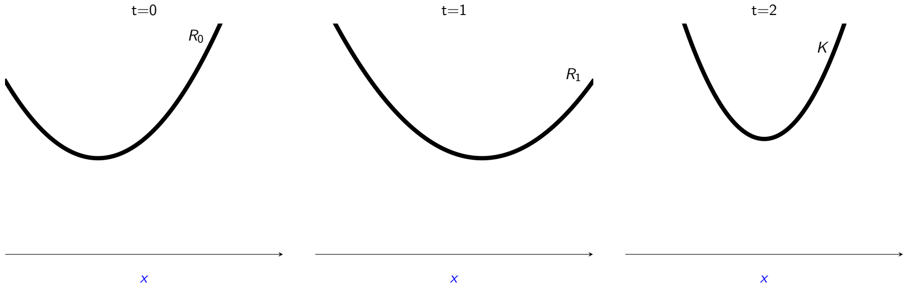
Final Cost $V_2 = K$

Abstract SDDP



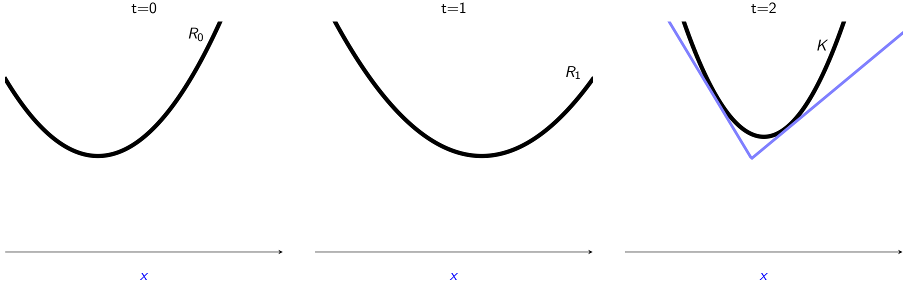
Real Bellman function $V_1 = T_1(V_2)$

Abstract SDDP



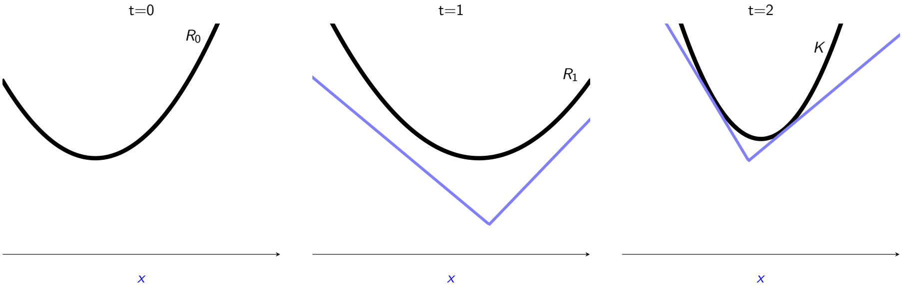
Real Bellman function $V_0 = T_0(V_1)$

Abstract SDDP



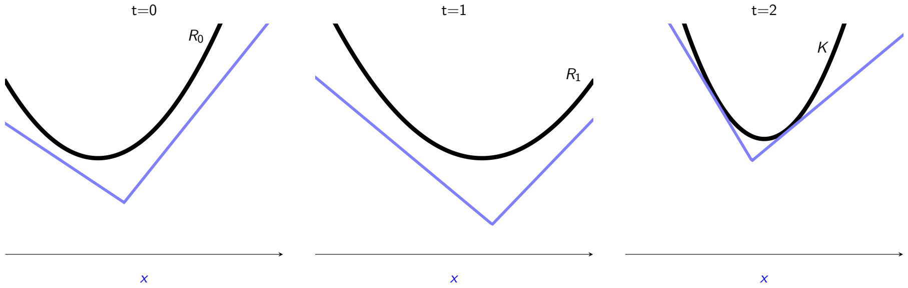
Lower polyhedral approximation $\underline{K}$ of K

Abstract SDDP



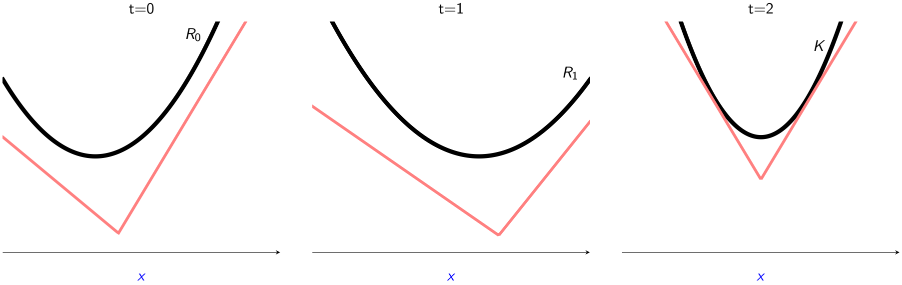
Lower polyhedral approximation $\underline{V}_1 = T_t(\underline{K})$ of V_1

Abstract SDDP



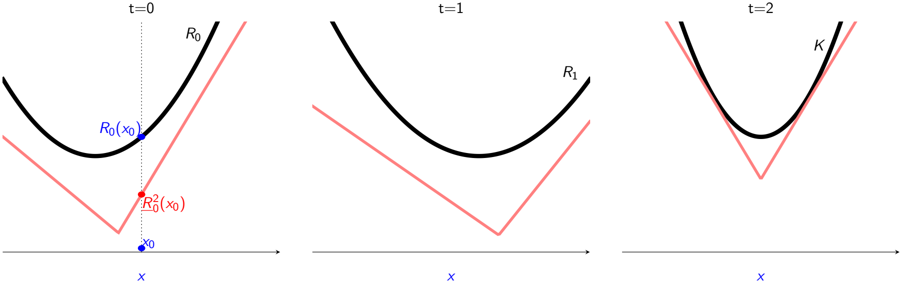
Lower polyhedral approximation $\underline{V}_0 = \mathcal{T}_t(\underline{V}_1)$ of V_0

Abstract SDDP



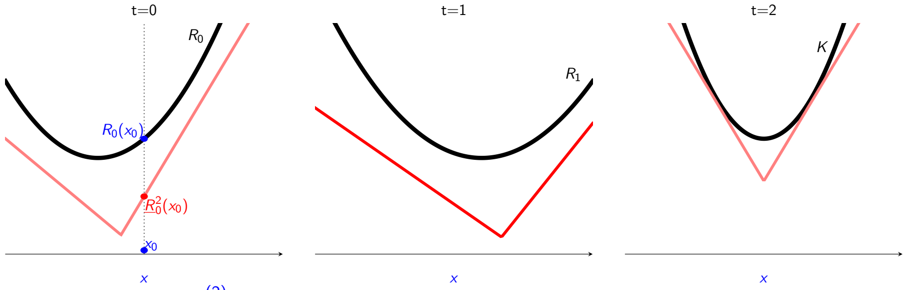
Assume that we have lower polyhedral approximations of V_t

Abstract SDDP



Thus we have a lower bound on the value of our problem

Abstract SDDP



We apply $\pi_0^{\underline{V}_1^{(2)}}$ to x_0 and obtain $\boldsymbol{X}_1^{(2)}$

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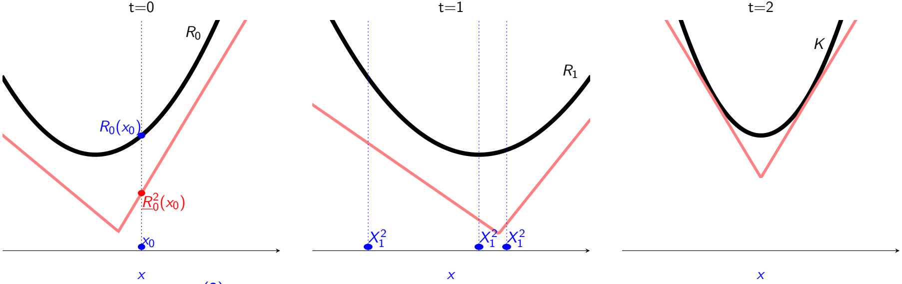

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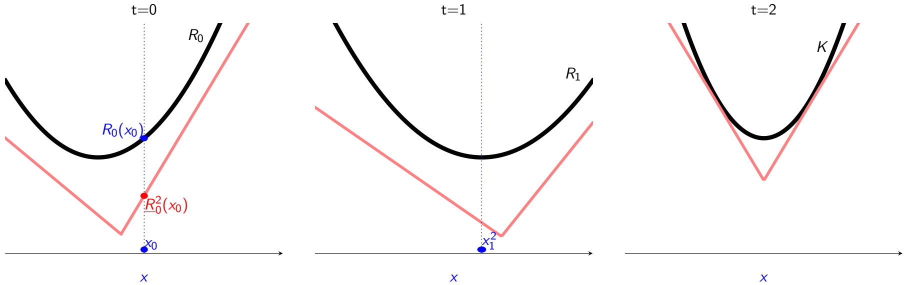
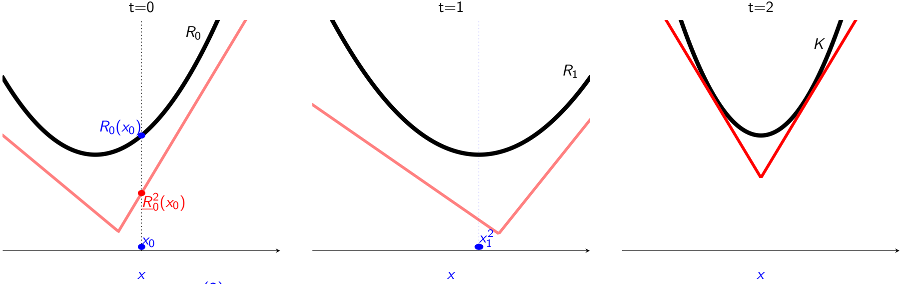


Draw a random realisation $x_1^{(2)}$ of $\boldsymbol{X}_1^{(2)}$

Abstract SDDP



We apply $\pi_1^{\underline{V}_1^{(2)}}$ to $x_1^{(2)}$ and obtain $\boldsymbol{X}_2^{(2)}$

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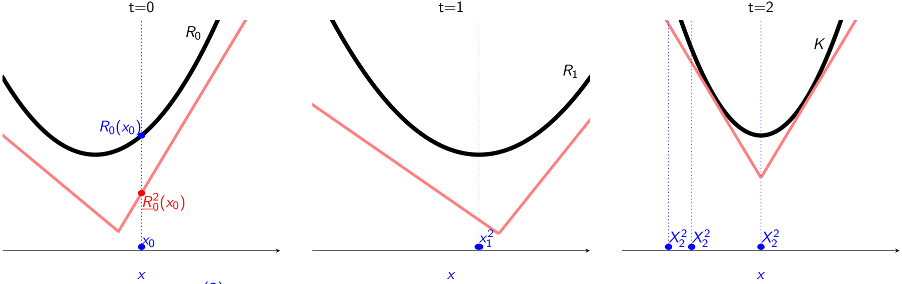

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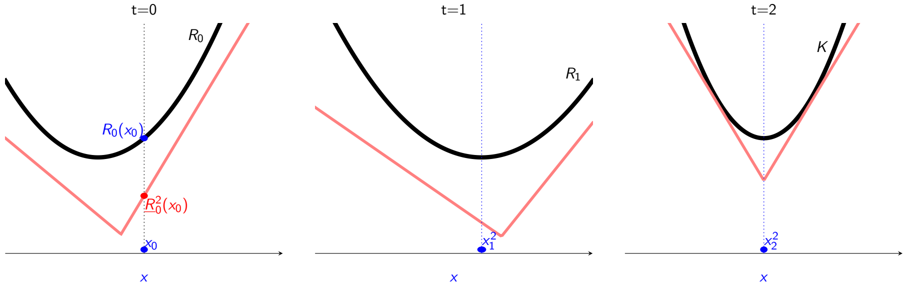
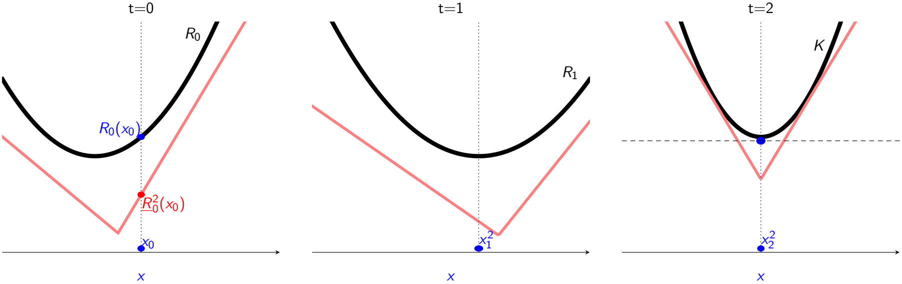


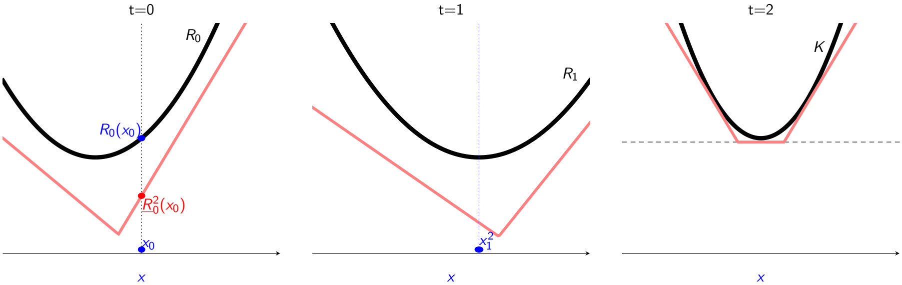
Draw a random realisation $x_2^{(2)}$ of $\boldsymbol{X}_2^{(2)}$

Abstract SDDP



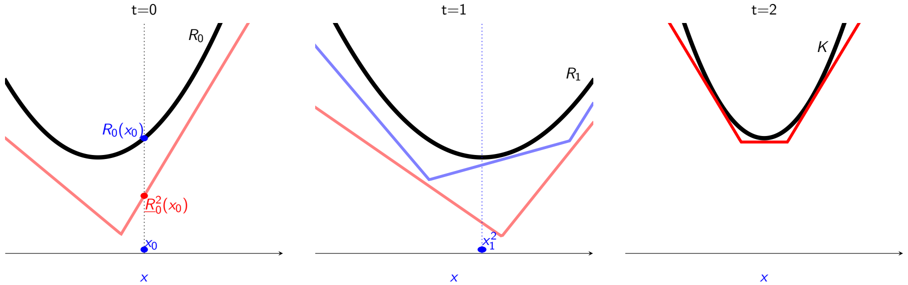
Compute a cut for K at $x_2^{(2)}$

Abstract SDDP



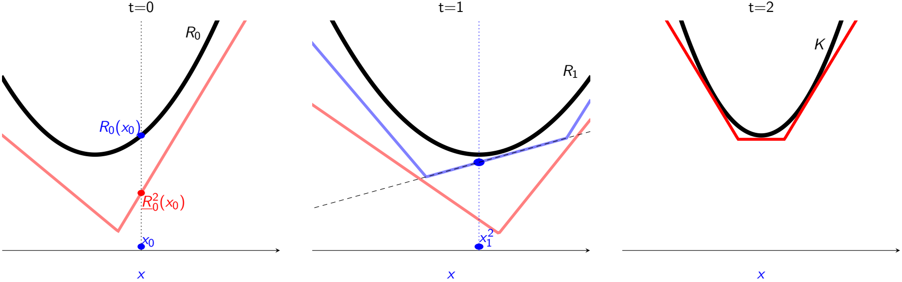
Add the cut to $\underline{V}_2^{(2)}$ which gives $\underline{V}_2^{(3)}$

Abstract SDDP



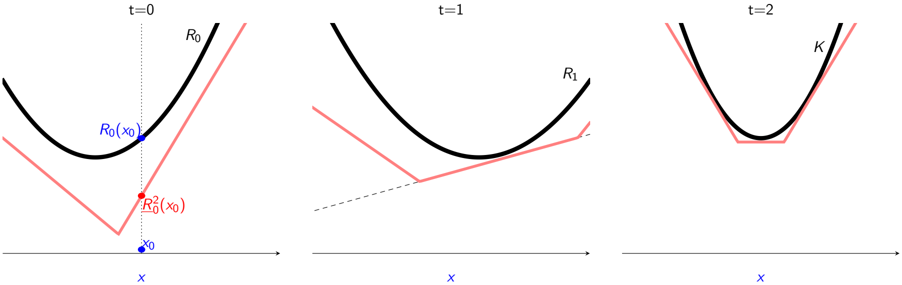
A new lower approximation of V_1 is $T_1(\underline{V}_2^{(3)})$

Abstract SDDP



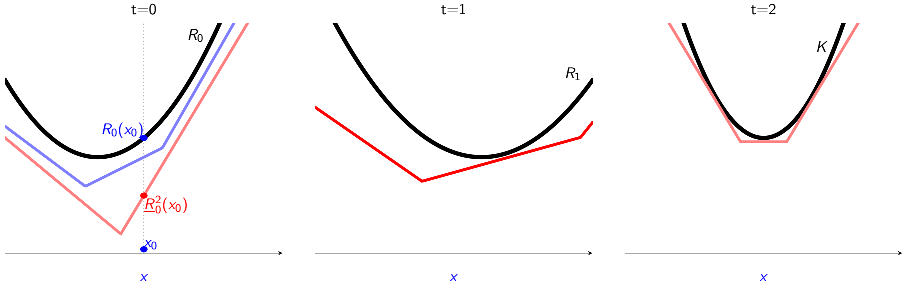
We only compute the face active at $x_1^{(2)}$

Abstract SDDP



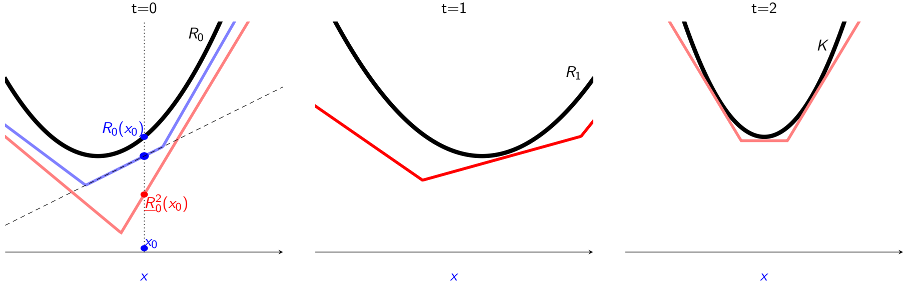
Add the cut to $\underline{V}_1^{(2)}$ which gives $\underline{V}_1^{(3)}$

Abstract SDDP



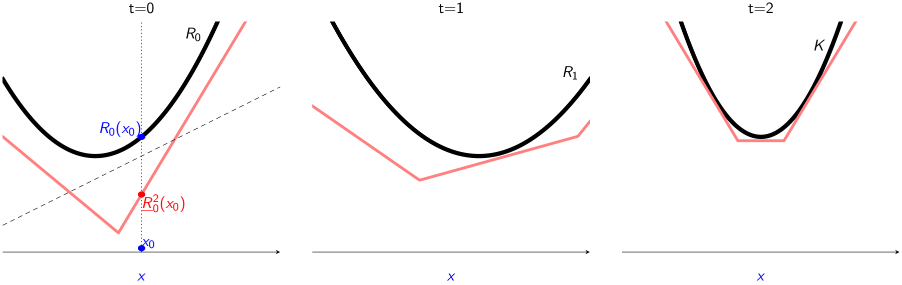
A new lower approximation of V_0 is $T_0(\underline{V}_1^{(3)})$

Abstract SDDP



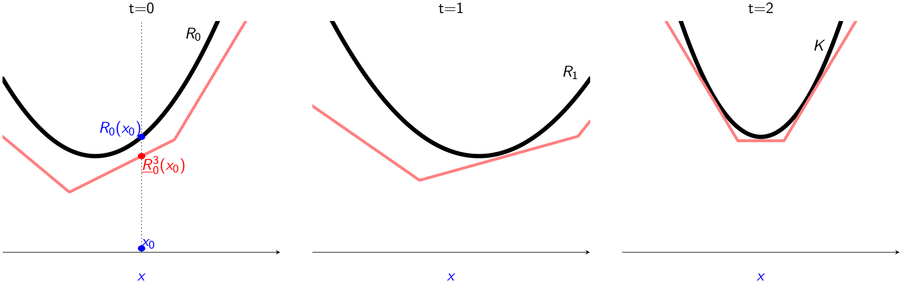
We only compute the face active at x_0

Abstract SDDP



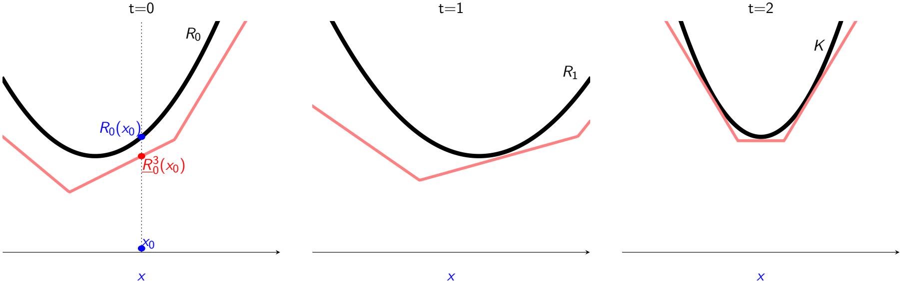
We only compute the face active at x_0

Abstract SDDP



We obtain a new lower bound

Abstract SDDP



We obtain a new lower bound

Data: Initial point x_0
$\underline{R}_t^{(0)} \leftarrow -\infty$
for $k : 0,\ 1, \dots$ **do**
// Forward Pass : compute a set of trial points $\{x_t^k\}_{t\in[\![0,T]\!]}$
$x_0^k \leftarrow x_0$
for $t : 0$ **to** $T-1$ **do**
select $\boldsymbol{x}_{t+1}^k \in \arg\min \mathcal{T}_t(\underline{R}_{t+1}^k)(x_t^k)$
Randomly select $\omega_t^k \in \Omega$
$x_{t+1}^k \leftarrow \boldsymbol{x}_{t+1}^k(\omega_t^k)$
end for
// Backward Pass : refine the lower approximations at the trial points
$\underline{R}_T^{k+1} \leftarrow K$
for $t : T-1$ **to** 0 **do**
$\theta_t^{k+1} \leftarrow \mathcal{B}_t(\underline{R}_{t+1}^{k+1})(x_t^k)$ // cut coefficients
select $\lambda_t^{k+1} \in \partial\mathcal{B}_t(\underline{R}_{t+1}^{k+1})(x_t^k)$
$\beta_t^{k+1} \leftarrow \theta_t^{k+1} - \langle \lambda_t^{k+1}, \overline{x}_t^k \rangle$
Define $\mathcal{C}_t^{k+1} : x \mapsto \langle \lambda_t^{k+1}, x \rangle + \beta_t^{k+1}$ // new cut
$\underline{R}_t^{k+1} \leftarrow \max\{\underline{R}_t^k, \mathcal{C}_t^{k+1}\}$ // update lower approximation
end for
STOP if some stopping test is satisfied
end for

Absract SDDP convergence

Theorem

Assume that Ω is finite, $R(x_0)$ is finite, and $\{\mathcal{B}_t\}_t$ is compatible. Further assume that, for all $t \in [\![0, T]\!]$ there exists compact sets X_t such that, for all k, $x_t^k \in X_t$ (e.g. $\mathcal{B}_t$ have compact domain).

Then, $(\underline{R}_t^k)_{k\in\mathbb{N}}$ is a non-decreasing sequence of lower approximations of R_t, and $\lim_k \underline{R}_0^k(x_0) = R_0(x_0)$, for $t \in [\![0, T-1]\!]$.

Further, the cuts coefficients generated remain in a compact set.

Contents

Stochastic Optimization problem

Recall the optimization problem

$$\begin{aligned} \min_{\pi} \quad & \mathbb{E}\Big[\sum_{t=0}^{T-1} L_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{\xi}_t) + K(\boldsymbol{X}_T)\Big] \\ s.t. \quad & \boldsymbol{X}_{t+1} = f_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{\xi}_t) \\ & \boldsymbol{U}_t = \pi_t(\boldsymbol{X}_t, \boldsymbol{\xi}_t) \in U_t(x, \xi) \end{aligned}$$

With associated Dynamic Programming equation

$$\begin{cases} V_T(x) & = \quad K(x) \\ \hat{V}_t(x, \xi) & = \quad \min_{u_t \in U_t(x,\xi)} L_t(x, u_t, \xi) + V_{t+1} \circ f_t(x, u_t, \xi) \\ V_t(x) & = \quad \mathbb{E}\Big[\hat{V}_t(x, \boldsymbol{\xi}_t)\Big] \end{cases}$$

Primal Bellman equation

Consequently we introduce the following Bellman operators

$$\hat{\mathcal{T}}_t(R)(x,\xi) = \min_{u_t \in \mathbb{U}} L_t(x,u_t,\xi) + R \circ f_t(x,u_t,\xi)$$

and

$$\mathcal{T}_t(R) : x \mapsto \mathbb{E}\big[\hat{\mathcal{T}}_t(R(x,\boldsymbol{\xi}_t))\big]$$

Which allow to rewrite the Dynamic Programming equation as

$$\begin{cases} V_T &= K \\ \hat{V}_t &= \hat{\mathcal{T}}_t(V_{t+1}) \\ V_t &= \mathcal{T}_t(V_{t+1}) \end{cases}$$

Contents

What are the specificities of Primal SDDP algorithm ?

- In the forward pass you don't need to solve $\mathcal{T}_t(\underline{V}_{t+1}^k)(x_t^k)$
- It is enough to solve $\hat{\mathcal{T}}_t(\underline{V}_{t+1}^k)(x_t^k, \xi_t^k)$
- In the backward pass you need to solve $\hat{\mathcal{T}}_t(\underline{V}_{t+1}^{k+1})(x_t^k, \xi)$ for all $\xi \in \text{supp}(\boldsymbol{\xi}_t)$
- And the cut coefficients are computed as the mean of the cut coefficient associated to $\hat{V}_t$
- And that's it !

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- And that's it !

Data: initial point x_0, initial lower bounds $\underline{V}_t^0$ on V_t
for $k : 0, 1, \dots$ **do**
// Forward Pass : compute a set of trial points $\{x_t^k\}_{t \in [\![0,T]\!]}$
Draw a noise scenario $\{\xi_t^k\}_{t \in [\![1,T]\!]}$
for $t : 0$ **to** $T-1$ **do**
select $x_{t+1}^k \in \arg\min \widehat{\mathcal{T}}_t(\underline{V}_{t+1}^k)(x_t^k, \xi_{t+1}^k)$
end for
// Backward Pass : refine the lower-approximations at the trial points
for $t : T-1$ **to** 0 **do**
for $\xi \in \mathrm{supp}(\boldsymbol{\xi}_{t+1})$ **do**
$\underline{\theta}_t^{\xi,k+1} \leftarrow \hat{\mathcal{T}}_t(\underline{V}_{t+1}^{k+1})(x_t^k, \xi)$
$\underline{\lambda}_t^{\xi,k+1} \in \partial \hat{\mathcal{T}}_t(\underline{V}_{t+1}^{k+1})(x_t^k, \xi)$
end for
$$\underline{\lambda}_t^{k+1} \leftarrow \sum_{\xi \in \mathrm{supp}(\boldsymbol{\xi}_{t+1})} \pi_{t+1}^{\xi} \underline{\lambda}_t^{\xi,k+1}$$
$$\underline{\beta}_t^{k+1} \leftarrow \sum_{\xi \in \mathrm{supp}(\boldsymbol{\xi}_{t+1})} \pi_{t+1}^{\xi} \big(\underline{\theta}_t^{\xi,k+1} - \langle \underline{\lambda}_t^{\xi,k+1}, x_t^k \rangle\big)$$
$\underline{V}_t^{k+1} \leftarrow \max\{\underline{V}_t^k, \langle \underline{\lambda}_t^k, \cdot \rangle + \underline{\beta}_t^{k+1}\}$ // update lower approximation
end for
STOP if some stopping test is satisfied
end for

Contents

Fenchel transform of LBO

Theorem

Assume that the pair $(\mathcal{B}, R)$ satisfy the RCR assumption, R being proper polyhedral, and $\mathcal{B}$ compact (i.e. $\mathcal{G}$ is compact valued with compact domain).

Then $\mathcal{B}(R)$ is a proper function and we have that

$$[\mathcal{B}(R)]^\star = \mathcal{B}^\ddagger(R^\star)$$

where $\mathcal{B}^\ddagger$ is an explicitely given LBO.

Dual LBO

More precisely we have

$$\begin{aligned}
\mathcal{B}^{\ddagger}(Q) : \lambda \mapsto \inf_{\boldsymbol{\mu} \in \mathcal{L}^0(\mathbb{R}^{n_x}), \boldsymbol{\nu} \in \mathcal{L}^0(\mathbb{R}^{n_c})} \quad & \mathbb{E}\Big[-\boldsymbol{\mu}^\top \boldsymbol{h} + Q(\boldsymbol{\nu}) \Big] \\
s.t. \quad & T^\top \mathbb{E}\big[\boldsymbol{\mu}\big] + \lambda = 0 \\
& \mathcal{W}_u^{\dagger}(\boldsymbol{\mu}) = \boldsymbol{C} \\
& \mathcal{W}_y^{\dagger}(\boldsymbol{\mu}) = \boldsymbol{\nu} \\
& \boldsymbol{\mu} \leq 0
\end{aligned}$$

Contents

Recursion over dual value function

Denote $\mathcal{D}_t := V_t^\star$.

Theorem

Then

$$\begin{cases} \mathcal{D}_T &= K^\star \\ \mathcal{D}_t &= \mathcal{B}^\ddagger_{t,L_{t+1}}(\mathcal{D}_{t+1}) \qquad \forall t \in [\![0, T-1]\!] \end{cases}$$

where $\mathcal{B}^\ddagger_{t,L_{t+1}} := \mathcal{B}^\ddagger_t + \mathbb{I}_{\|\lambda_{t+1}\|_\infty \leq L_{t+1}}$.

This is a Bellman recursion on $\mathcal{D}_t$ instead of V_t.
Further, under easy technical assumptions, $\left\{\mathcal{B}^\ddagger_{t,L_{t+1}}\right\}_{t\in[\![0,T]\!]}$ is a compatible sequence of LBOs, where V_t is L_t-Lipschitz.

What's different with Dual SDDP

- It's exactly the abstract SDDP algorithm applied to the dual Bellman Operator...
- except that we need a starting point, and some bounds on the dual variables.
- the main differences with primal SDDP:
  - need to solve the coupled problem $\mathcal{T}_t^{\ddagger}(\mathcal{D}_{t+1}^k)(\lambda_t^k)$ in the forward pass, instead of a "one-realisation" (hat) version
  - also need to solve the coupled problem in the Backward phase, instead of multiple "one-realisation" version that are averaged

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Data: Initial primal point x_0, Lipschitz bounds $\{L_t\}_{t\in[\![0,T]\!]}$

$\underline{\mathcal{D}}_t^{(0)} \leftarrow -\infty$

for $k : 0,\ 1, \dots$ **do**

// Forward Pass : compute a set of trial points $\{\lambda_t^{(k)}\}_{t\in[\![0,T]\!]}$

Select $\lambda_0^k \in \arg\max_{\|\lambda_0\|_\infty \le L_0} \Big\{ x_0^\top \lambda_0 - \underline{\mathcal{D}}_0^k(\lambda_0) \Big\}$ // Fenchel transform

for $t : 0$ **to** $T-1$ **do**

select $\boldsymbol{\lambda}_{t+1}^k \in \arg\min \mathcal{T}_{t,L_{t+1}}^{\ddagger}(\underline{\mathcal{D}}_{t+1}^k)(\lambda_t^k)$

and draw a realization λ_{t+1}^k of $\boldsymbol{\lambda}_{t+1}^k$

end for

// Backward Pass : refine the lower-approximations at the trial points

$\underline{\mathcal{D}}_T^k \leftarrow K^\star$.

for $t : T-1$ **to** 0 **do**

$\overline{\theta}_t^{k+1} \leftarrow \mathcal{T}_{t,L_{t+1}}^{\ddagger}(\underline{\mathcal{D}}_{t+1}^{k+1})(\lambda_t^k)$ // computing cut coefficients

select $\overline{x}_t^{k+1} \in \partial \mathcal{T}_{t,L_{t+1}}^{\ddagger}(\underline{\mathcal{D}}_{t+1}^{k+1})(\lambda_t^k)$

$\overline{\beta}_t^{k+1} \leftarrow \overline{\theta}_t^{k+1} - \langle \lambda_t^k , \overline{x}_t^{k+1} \rangle$

Define $\mathcal{C}_t^{k+1} : \lambda \mapsto \langle \overline{x}_t^{k+1} , \lambda \rangle + \overline{\beta}_t^{k+1}$

$\underline{\mathcal{D}}_t^{k+1} \leftarrow \max\{\underline{\mathcal{D}}_t^k, \mathcal{C}_t^{k+1}\}$ // update lower approximation

end for

STOP if some stopping test is satisfied

end for

Contents

Converging upper bound and stopping test

We have

$$\underline{V}_t^k \leq V_t$$

and

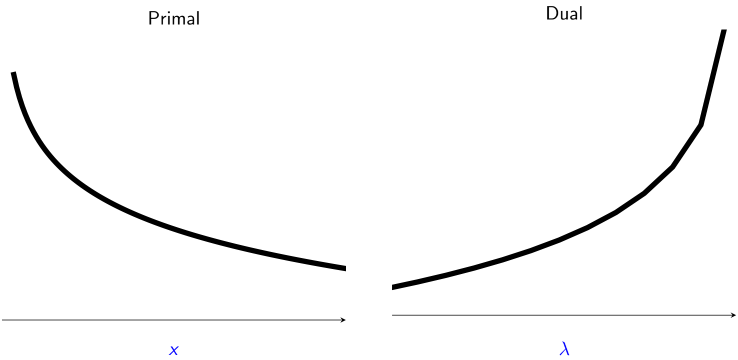
$$\underline{\mathcal{D}}_t^k \leq \mathcal{D}_t \quad \Longrightarrow \quad \underbrace{\left(\underline{\mathcal{D}}_t^k\right)^\star}_{:\approx \overline{V}_t^k} \geq \left(\mathcal{D}_t^\star\right) = V_t^{\star\star} = V_t$$

Finally, we obtain

$$\underline{V}_0(x_0) \leq V_0(x_0) \leq \overline{V}_0(x_0).$$

Using the convergence of the abstract SDDP algorithm we show that this bounds are converging, yielding converging deterministic stopping tests.

Link between primal and dual approximations



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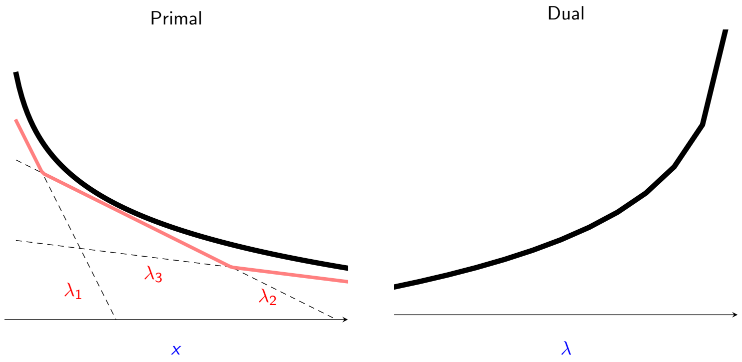

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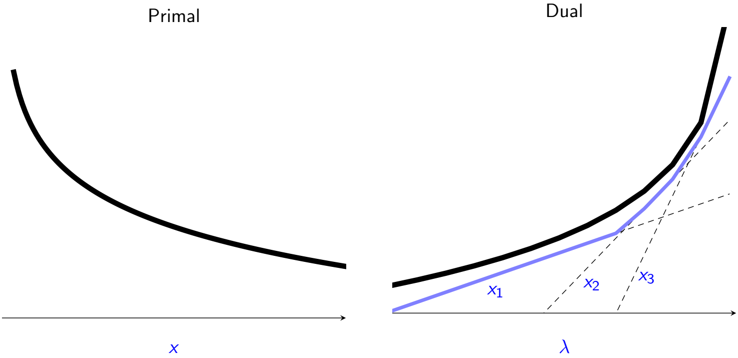

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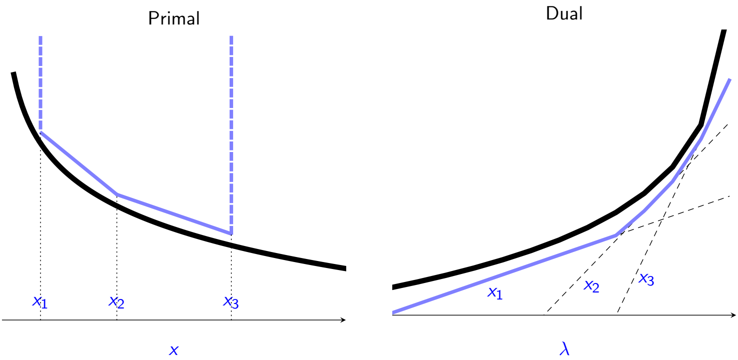

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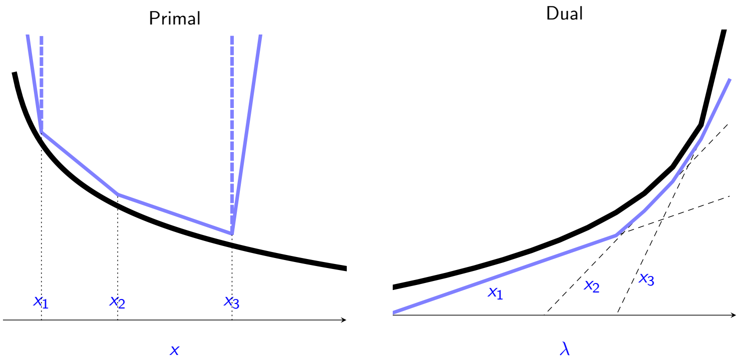


Contents

Inner Approximation

- $\overline{V}_t^k := \left[\underline{\mathcal{D}}_t^k\right]^\star$ which is lower than V_t on X_t
- Or

$$\overline{V}_t^k(x) = \min_{\sigma\in\Delta} \left\{ -\sum_{\kappa=1}^{k} \sigma_\kappa \overline{\beta}_t^\kappa \quad \Big| \quad \sum_{\kappa=1}^{k} \sigma_\kappa \overline{x}_t^\kappa = x \right\}$$

- The inner approximation can be computed by solving

$$\overline{V}_t^{k+1}(x) = \sup_{\lambda,\theta} \quad x^\top \lambda - \theta$$
$$s.t. \quad \theta \geq \left\langle \underline{x}_t^i \, , \lambda \right\rangle + \overline{\beta}_t^\kappa \qquad \forall \kappa \in [\![1, k]\!]$$

Inner Approximation - regularized

- $\overline{V}_t^k := \left[\underline{\mathcal{D}}_t^k\right]^\star \Box (L_t \| \cdot \|_1)$ which is lower than V_t on X_t
- Or

$$\overline{V}_t^k(x) = \min_{y \in \mathbb{R}^{n_x}, \sigma \in \Delta} \left\{ L_t \|x - y\|_1 - \sum_{\kappa=1}^k \sigma_\kappa \overline{\beta}_t^\kappa \quad \middle| \quad \sum_{\kappa=1}^k \sigma_\kappa \overline{x}_t^\kappa = y \right\}$$

- The inner approximation can be computed by solving

$$\begin{aligned} \overline{V}_t^{k+1}(x) = \sup_{\lambda,\theta} \quad & x^\top \lambda - \theta \\ s.t. \quad & \theta \geq \langle \underline{x}_t^i, \lambda \rangle + \overline{\beta}_t^\kappa \qquad \forall \kappa \in [\![1, k]\!] \\ & \|\lambda\|_\infty \leq L_t \end{aligned}$$

A converging strategy - with guaranteed payoff

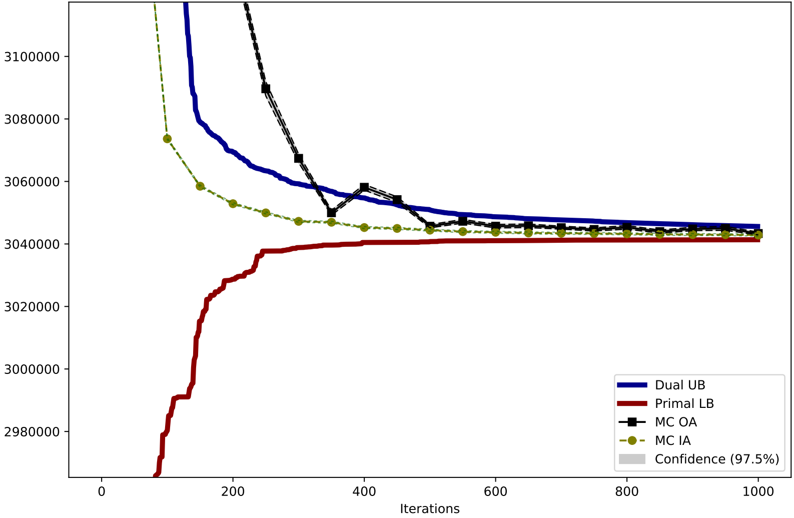
Theorem

Let $C_t^{IA,k}(x)$ be the expected cost of the strategy $\pi^{\overline{V}_t^k}$ when starting from state x at time t.
We have,

$$C_t^{IA,k}(x) \leq \overline{V}_t^k(x) \qquad \lim_k C_t^{IA,k}(x) = V_t(x)$$

Thus, the inner-approximation yields a new converging strategy, and we have an upper-bound on the (expected) value of this strategy.

Numerical results



Stopping test

	Dual stopping test		Statistical stopping test	
ε (%)	n it.	CPU time	n it.	CPU time
2.0	156	183s	250	618s
1.0	236	400s	300	787s
0.5	388	1116s	450	1429s
0.1	$>$ 1000	.	1000	5519s

Table: Comparing dual and statistical stopping criteria for different accuracy levels ε.

Conclusion

- We extend the SDDP algorithm to an abstract framework.
- Leveraging Fenchel conjugate we are able to show a dynamic recursion between dual Bellman value functions.
- We can apply SDDP to this dual recursion.
- This yields a converging exact upper bound on the value of the original problem, hence giving exact and converging stopping tests.
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More information : http://www.optimization-online.org/DB_FILE/2018/04/6575.pdf

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Implementation

- We developp an open-source software in the recent, fast and efficient julia language
- You can test Julia through www.juliabox.com
- This software focus on solving stochastic dynamic problem through dynamic programming and SDDP
- You can install the package in Julia simply by typing

```
Pkg.add("StochDynamicProgramming")
```

  in your Julia console
- You can test the dual version through https://github.com/frapac/DualSDDP.jl

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Anymore questions for me?