Robust Optimization Applied to Uncertain Limit Analysis

Jeremy Bleyer and Vincent Leclère

Abstract Limit analysis (LA) is an efficient tool for computing in a direct manner the ultimate load of a structure made of a perfectly plastic material. The lower bound static approach amounts to maximize the load factor such that one can find an optimal stress field in equilibrium with such loading and satisfying strength conditions at each point in the domain. In the deterministic case, the ultimate load is obtained via the resolution of a convex optimization problem. When loading or strength properties are random, the data of such an optimization problem become uncertain. Robust optimization theory is a branch of mathematical optimization which aim at finding an optimal solution of uncertain problems among all possible realizations of the uncertainty within a known uncertainty set. Applying the concepts of robust optimization to uncertain limit analysis, one may compute a worst-case ultimate load estimate associated with a given uncertainty set, for instance in the case of uncertain strength properties or uncertain load cases. This paper discusses how robust limit analysis problems can be reformulated, either exactly or approximately, into deterministic problems. In particular, the distinction between static and adjustable robust counterparts is introduced. In the former case, uncertain LA problems are replaced with a deterministic problem with reduced strength properties. In the latter case, additional optimization variables must be introduced in order to obtain an extended LA problem in much higher dimension.

Keywords Limit analysis · Ultimate load · Uncertainty · Robust optimization

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G. Garcea and D. Weichert (eds.), *Direct Methods for Limit State of Materials and Structures*, Lecture Notes in Applied and Computational Mechanics 101,
https://doi.org/10.1007/978-3-031-29122-7_11

1 Introduction

Limit analysis [13, 22] is a powerful direct method used to estimate the collapse load of a structure consisting of a perfectly plastic material. The lower and upper bound approaches of limit analysis are naturally formulated as convex optimization problems for which given data consist of a known material yield criterion, a known reference loading and a known geometry [12]. However, in real-world applications, these parameters may be subject to uncertainty due to factors such as inaccurate load amplitude or direction, or variations in material strength. As a result, engineers often aim to design structures that are robust to such uncertainties, meaning that the collapse load must be safe for all possible combinations of uncertain parameters.

Traditionally, limit analysis has addressed this issue by either assuming a worst-case scenario for the uncertain parameters or by performing a stochastic analysis in which random realizations of the parameters are used. While the first approach can be overly conservative, it can also be challenging to determine the worst-case scenario in complex loading situations. The second approach, on the other hand, requires assuming a probability distribution for the parameters and solving a large number of problems to find the worst-case configuration, which may not be achievable in practice. General definitions of the probability of collapse have been given in [3, 21], later revisited by [2] using stochastic stress vectors. Various works have also considered the numerical computation of limit loads in a stochastic setting such as [23, 28] or [1, 10, 14, 17] for geotechnical applications. For instance, the reader can refer to [15] for a recent review of slope stability in spatially variable soils.

Alternative approaches have sought to evaluate the robustness or reliability of structures through non-probabilistic methods. In [19], the authors consider uncertain limit analysis of truss structures with very similar sources of uncertainties as those investigated in this work. For this purpose, they used the info-gap decision theory [4] which is however known to be difficult to apply in practice since robustness functions are very hard to compute in general. For the very specific case of truss structures investigated in [19], it can however be computed via the resolution of a linear programming problem. Similarly, mixed-integer programming approaches can also be used to compute a worst-case limit load [16] but solving such NP-hard problems is notoriously difficult and almost impossible for large-scale problems. Using a chance-constrained programming approach, [25, 26] considered limit analysis and shakedown theorems under normal or log-normal strength uncertainties for von Mises plasticity.

In this work, we propose an alternative approach that utilizes the principles of robust optimization theory [5, 7] to obtain a robust estimate of plastic limit loads in the presence of uncertainty. This approach allows us to design structures that are resistant to a wide range of uncertain parameters without relying on conservative assumptions or computationally intensive analyses. More precisely, uncertain limit analysis problems are formulated in the case of uncertain strength properties. A definition of the worst-case limit load is given using concepts of robust optimization theory. Then, in order to obtain computationally tractable formulations, different decision rules are introduced, in particular so-called *static* and *affinely adjustable* formulations. Static concepts are then applied to the definition of robust strength conditions and illustrated on the case of a Mohr-Coulomb criterion with uncertain cohesion and friction angle. Finally, the resolution of robust limit analysis at the structure scale is discussed for the case of strength uncertainties and loading uncertainties.

The manuscript is organized as follows: Sect. 2 introduces robust formulations of limit analysis theory in the case of strength uncertainty; Sect. 3 details the derivation of tractable robust counterparts of uncertain strength constraints arising in the previous formulations; Sect. 4 is devoted to the resolution of robust limit analysis problems with a specific emphasis on the case of loading uncertainties and the corresponding affinely adjustable robust formulations; finally, Sect. 5 draws some conclusions and perspectives for future research.

2 Robust Limit Analysis with Strength Uncertainties

2.1 Nominal and Uncertain Limit Analysis Problem

The *nominal* limit analysis problem amounts to computing the maximum load factor λ_N by solving the following convex maximization problem:

$$\begin{aligned} \lambda_\mathrm{N} = \max_{\lambda,\boldsymbol{\sigma}} \;& \lambda \\ \text{s.t. } & \div\boldsymbol{\sigma} + \lambda \boldsymbol{f}^\mathrm{r} + \boldsymbol{f}^\mathrm{f} = 0 && \text{in } \Omega \\ & \boldsymbol{\sigma}\cdot\boldsymbol{n} = \lambda \boldsymbol{t}^\mathrm{r} + \boldsymbol{t}^\mathrm{f} && \text{on } \partial\Omega_T \\ & \boldsymbol{\sigma} \in G && \text{in } \Omega \end{aligned} \tag{N}$$

where λ is the load factor, $\boldsymbol{\sigma}$ the Cauchy stress field in Ω, $\boldsymbol{f}^\mathrm{r}$ (resp. $\boldsymbol{f}^\mathrm{f}$) is the reference (resp. fixed) body force, $\boldsymbol{t}^\mathrm{r}$ (resp. $\boldsymbol{t}^\mathrm{f}$) the reference (resp. fixed) contact force prescribed on some part $\partial\Omega_T$ of the boundary with unit normal $\boldsymbol{n}$ and G is the material yield/strength criterion which we assume to be a convex set (possibly unbounded) containing 0. In the above, the first two constraints correspond to the local balance equation and traction boundary conditions, whereas the last one corresponds to the strength condition which must be satisfied at all points $\boldsymbol{x} \in \Omega$. Note that formulation (N) corresponds to a static formulation which will result in a lower-bound estimate of the true collapse load when restricting to a finite-element subspace of statically admissible stress fields.

We now consider the case where the loading is certain but the material may possess uncertain properties such that the strength criterion is now written as $G(\boldsymbol{\zeta})$ where $\boldsymbol{\zeta} \in \mathbb{R}^m$ is a vector of uncertain parameters. Contrary to probabilistic approaches in which $\boldsymbol{\zeta}$ is a random variable with a given probability distribution, robust optimization approaches describe the uncertainty through the notion of an *uncertainty set* $\mathcal{U} \subseteq$

$\mathbb{R}^m$. It is assumed that any possible realization of the uncertainty belongs to the uncertainty set $\boldsymbol{\zeta} \in \mathcal{U}$ without positing any probability distribution. The goal of robust optimization theory is to find an optimum solution to an uncertain optimization problem for any possible realization in this uncertainty set. Obviously, the choice of the uncertainty set is an important modeling step in such approaches and depends on our knowledge of the origins of the considered uncertainty. If probability distributions are known, uncertainty sets can be based on the size of the support or the shape of the probability distribution. For instance, its size can correspond to a certain confidence level of the probability distribution. It can also be built from available data.

This aspect is outside the scope of the present work, which presents a general methodology. One key assumption on the uncertainty used to obtain interesting results is that it is assumed to be convex. Although it can be more general, we assume, for simplicity, that $\mathcal{U}$ is a convex ball of unit radius for some norm i.e. $\mathcal{U} = \{\boldsymbol{\zeta} \in \mathbb{R}^m \text{ s.t. } \|\boldsymbol{\zeta}\| \leq 1\}$. In particular, we will note by $\mathcal{U}_p$ uncertainty sets corresponding to the L_p-ball (typically with $p = 1, 2$ or ∞).

The maximum load factor now becomes uncertain i.e. it depends on the value $\boldsymbol{\zeta}$ of the uncertainty realization:

$$\begin{aligned} \lambda^+(\boldsymbol{\zeta}) = \max_{\lambda,\boldsymbol{\sigma}} \ & \lambda \\ \text{s.t. } & \div \boldsymbol{\sigma} + \lambda \boldsymbol{f}^{\mathrm{r}} + \boldsymbol{f}^{\mathrm{f}} = 0 \\ & \boldsymbol{\sigma} \cdot \boldsymbol{n} = \lambda \boldsymbol{t}^{\mathrm{r}} + \boldsymbol{t}^{\mathrm{f}} \\ & \boldsymbol{\sigma} \in G(\boldsymbol{\zeta}) \end{aligned} \tag{1}$$

The main purpose of robust optimization is to provide worst-case solutions to a given optimization problem. Our proposed theory of robust limit analysis therefore aims at evaluating the worst-case limit load among all possible realizations. In the remaining of this section, we discuss various robust formulations.

2.2 *Adjustable Robust Optimization*

For a given loading and two different given realizations of the uncertainty, one expects that the corresponding optimal stress fields will be different depending on the uncertainty realizations. The most natural approach therefore consists in considering the stress field and the corresponding load factor to be *recourse variables*, i.e. variables which depends on $\boldsymbol{\zeta}$. Thus, we are faced with an *adjustable robust counterpart* (ARC) to problem (1) defined as follows:

$$\begin{aligned} \lambda_{\mathrm{ARC}} = \min_{\boldsymbol{\zeta}\in\mathcal{U}} \lambda^+(\boldsymbol{\zeta}) = \min_{\boldsymbol{\zeta}\in\mathcal{U}} \max_{\boldsymbol{\sigma}(\boldsymbol{\zeta}),\lambda(\boldsymbol{\zeta})} \ & \lambda(\boldsymbol{\zeta}) \\ \text{s.t. } & \div \boldsymbol{\sigma}(\boldsymbol{\zeta}) + \lambda(\boldsymbol{\zeta}) \boldsymbol{f}^{\mathrm{r}} + \boldsymbol{f}^{\mathrm{f}} = 0 \\ & \boldsymbol{\sigma}(\boldsymbol{\zeta}) \cdot \boldsymbol{n} = \lambda(\boldsymbol{\zeta}) \boldsymbol{t}^{\mathrm{r}} + \boldsymbol{t}^{\mathrm{f}} \\ & \boldsymbol{\sigma}(\boldsymbol{\zeta}) \in G(\boldsymbol{\zeta}) \end{aligned} \tag{ARC}$$

i.e. we find the largest load factor such that, for each uncertainty realization there exists an optimal stress field in equilibrium, with the corresponding collapse load factor, satisfying the strength criterion.

In the following, we also make use of the following equivalent formulation of the ARC problem [18, 24]:

$$\begin{aligned}\lambda_{\text{ARC}} = \max_{\bar{\lambda}} \ & \bar{\lambda} \\ \text{s.t. } & \forall \boldsymbol{\zeta} \in \mathcal{U}, \exists \boldsymbol{\sigma}, \lambda \text{ s.t. } \div \boldsymbol{\sigma} + \lambda \boldsymbol{f}^{\text{r}} + \boldsymbol{f}^{\text{f}} = 0 \\ & \qquad\qquad\qquad\quad \boldsymbol{\sigma} \cdot \boldsymbol{n} = \lambda \boldsymbol{t}^{\text{r}} + \boldsymbol{t}^{\text{f}} \\ & \qquad\qquad\qquad\quad \boldsymbol{\sigma} \in G(\boldsymbol{\zeta}) \\ & \qquad\qquad\qquad\quad \bar{\lambda} \leq \lambda \end{aligned} \tag{2}$$

where uncertainty of the objective function has been transferred to the constraints with the introduction of a static (non-adjustable) variable $\bar{\lambda}$.

2.3 Static Robust Optimization

Unfortunately, adjustable recourse problems are numerically challenging. Indeed, both formulations involve either a min/max problem (ARC) or an infinite number of constraints (2). To solve adjustable recourse problem, one typically makes a simplifying assumption on how recourse variables depend on the uncertainty, the so-called *decision rules*.

The most simple of such rules is to assume that recourse variables are in fact static, i.e. they do not depend on the uncertainty. This yields to a conservative static robust counterpart (RC) in which we look for a stress field $\boldsymbol{\sigma}$ and a load factor λ, independent of the exact realization of the uncertainty, which satisfy the strength condition $G(\boldsymbol{\zeta})$ for all $\boldsymbol{\zeta} \in \mathcal{U}$. The corresponding problem can be formulated as follows:

$$\begin{aligned}\lambda_{\text{RC}} = \max_{\lambda, \boldsymbol{\sigma}} \ & \lambda \\ \text{s.t. } & \div \boldsymbol{\sigma} + \lambda \boldsymbol{f}^{\text{r}} + \boldsymbol{f}^{\text{f}} = 0 \\ & \boldsymbol{\sigma} \cdot \boldsymbol{n} = \lambda \boldsymbol{t}^{\text{r}} + \boldsymbol{t}^{\text{f}} \\ & \boldsymbol{\sigma} \in G(\boldsymbol{\zeta}) \qquad\qquad \forall \boldsymbol{\zeta} \in \mathcal{U} \end{aligned} \tag{3}$$

What makes problem (3) a *robust optimization* problem is the condition $\forall \boldsymbol{\zeta} \in \mathcal{U}$ in the last constraint. This implies that the constraint $\boldsymbol{\sigma} \in G(\boldsymbol{\zeta})$ must be fulfilled for any possible value of $\boldsymbol{\zeta} \in \mathcal{U}$. It is therefore an infinite-dimensional constraint. One of the main goals of robust optimization theory is to make such a problem tractable using standard convex optimization algorithms.

For instance, the robust constraint can be reformulated as:

$$\boldsymbol{\sigma} \in G(\boldsymbol{\zeta}) \quad \forall \boldsymbol{\zeta} \in \mathcal{U} \quad \Leftrightarrow \quad \boldsymbol{\sigma} \in G_{\text{RC}} \tag{4}$$

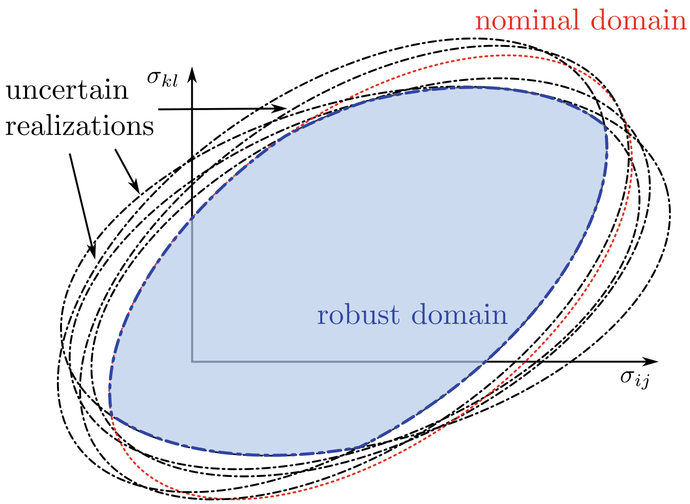


Fig. 1 Robust strength domain G_{RC} (in blue) obtained as the intersection of various uncertain realizations $G(\boldsymbol{\zeta})$ (in black) of a nominal domain (in red)

when introducing:

$$G_{\text{RC}} = \bigcap_{\boldsymbol{\zeta}\in\mathcal{U}} G(\boldsymbol{\zeta}) \tag{5}$$

the robust counterpart to the uncertain strength criterion. In order for a stress field to be admissible with respect to any possible realization of the uncertain strength criterion $G(\boldsymbol{\zeta})$, it has to belong to the intersection of all such domains (see Fig. 1).

Now, problem (3) writes as:

$$\begin{aligned} \lambda_{\text{RC}} = \max_{\lambda,\boldsymbol{\sigma}}\ & \lambda \\ \text{s.t.}\ & \div\boldsymbol{\sigma} + \lambda\boldsymbol{f}^{\text{r}} + \boldsymbol{f}^{\text{f}} = 0 \\ & \boldsymbol{\sigma}\cdot\boldsymbol{n} = \lambda\boldsymbol{t}^{\text{r}} + \boldsymbol{t}^{\text{f}} \\ & \boldsymbol{\sigma} \in G_{\text{RC}} \end{aligned} \tag{RC}$$

which is now independent of the uncertainty realization. As a result, problem (RC) is a classical limit analysis problem with a different strength criterion given by (5). This makes problem (RC) very appealing provided that a simple expression for G_{RC} can be found. It is however very hard to determine a simple expression for the infinite-dimensional set intersection appearing in (5). Exact or approximate reformulation of strength criteria robust counterparts are discussed in Sect. 3.

2.4 Affinely Adjustable Robust Optimization

Unfortunately, if (RC) problems are numerically tractable, the obtained approximation might be unreasonably conservative [8]. A middle ground is the *affinely adjustable robust counterpart* (AARC), which consists in looking for adjustable variables $\boldsymbol{\sigma}(\boldsymbol{\zeta})$ and $\lambda(\boldsymbol{\zeta})$ that are affine functions of the uncertain variable, the so-called *affine decision rule* [6]:

$$\boldsymbol{\sigma}(\boldsymbol{\zeta}) = \boldsymbol{\sigma}_0 + \sum_{j=1}^{m} \boldsymbol{\sigma}_j \zeta_j \tag{6a}$$

$$\lambda(\boldsymbol{\zeta}) = \lambda_0 + \sum_{j=1}^{m} \lambda_j \zeta_j \tag{6b}$$

where the $\boldsymbol{\sigma}_i$ (resp. λ_i) represent $1 + m$ different stress fields (load factor variables) which are now static optimization variables. Inserting the affine decision rules (6a)–(6b) into ($\texttt{ARC}$), the corresponding AARC reads:

$$\begin{aligned} \lambda_{\text{AARC}} = \max_{\boldsymbol{\sigma}_i,\lambda_i} \min_{\boldsymbol{\zeta}\in\mathcal{U}} \; & \lambda_0 + \sum_{j=1}^{m} \lambda_j \zeta_j \\ \text{s.t.} \; & \div\left(\boldsymbol{\sigma}_0 + \sum_{j=1}^{m} \boldsymbol{\sigma}_j \zeta_j\right) + \left(\lambda_0 + \sum_{j=1}^{m} \lambda_j \zeta_j\right) \boldsymbol{f}^{\text{r}} + \boldsymbol{f}^{\text{f}} = 0 \\ & \left(\boldsymbol{\sigma}_0 + \sum_{j=1}^{m} \boldsymbol{\sigma}_j \zeta_j\right) \cdot \boldsymbol{n} = \left(\lambda_0 + \sum_{j=1}^{m} \lambda_j \zeta_j\right) \boldsymbol{t}^{\text{r}} + \boldsymbol{t}^{\text{f}} \\ & \left(\boldsymbol{\sigma}_0 + \sum_{j=1}^{m} \boldsymbol{\sigma}_j \zeta_j\right) \in G(\boldsymbol{\zeta}) \end{aligned} \tag{7}$$

which can also be reformulated as follows:

$$\begin{aligned} \lambda_{\text{AARC}} = \max_{\bar{\lambda},\boldsymbol{\sigma}_i,\lambda_i} \; & \bar{\lambda} \\ \text{s.t.} \; & \div(\boldsymbol{\sigma}_j) + \lambda_j \boldsymbol{f}^{\text{r}} + \boldsymbol{f}^{\text{f}} = 0 \quad && \forall j = 0, \dots, m \\ & \boldsymbol{\sigma}_j \cdot \boldsymbol{n} = \lambda_j \boldsymbol{t}^{\text{r}} + \boldsymbol{t}^{\text{f}} && \forall j = 0, \dots, m \\ & \left(\boldsymbol{\sigma}_0 + \sum_{j=1}^{m} \boldsymbol{\sigma}_j \zeta_j\right) \in G(\boldsymbol{\zeta}) \; \forall \boldsymbol{\zeta} \in \mathcal{U} \\ & \bar{\lambda} \le \lambda_0 + \sum_{j=1}^{m} \lambda_j \zeta_j && \forall \boldsymbol{\zeta} \in \mathcal{U} \end{aligned} \tag{AARC}$$

in which we removed the uncertainty from the objective function and replaced the minimization over $\boldsymbol{\zeta}$ with robust constraints. Note that equality constraints depending on $\boldsymbol{\zeta}$ have been re-expressed by identifying the corresponding terms of the expansion in terms of ζ_i since $\mathcal{U}$ is full dimensional.

2.5 Comparison Between the Different Approaches

Summarizing, (RC) is the most conservative formulation yielding the smallest limit load. (AARC) is more flexible since it considers additional static variables $\boldsymbol{\sigma}_j, \lambda_j$ for $j = 1, \dots, m$ and reduces to (RC) if we fix all $\boldsymbol{\sigma}_j = 0$. As mentioned, (ARC) is less conservative than (AARC) since we allow for more general decision rules but is generally untractable. Finally, all of these formulations guard against all possible realizations of the uncertainty such that we have the following ordering:

$$\lambda_{\text{RC}} \leq \lambda_{\text{AARC}} \leq \lambda_{\text{ARC}} \leq \lambda^+(\boldsymbol{\zeta}) \quad \forall \boldsymbol{\zeta} \in \mathcal{U} \tag{8}$$

In the remainder of this work, the focus is put on the tractability of the different formulations. For (RC) to be tractable, the characterization of the safe domain G_{RC} must be tractable. Section 3 discusses conditions for which exact or approximate tractable formulations can be obtained. Tractable formulations of (AARC) are then discussed in Sect. 4.

3 Robust Strength Conditions

3.1 Uncertain Strength Conditions and a Tractable Approximation

Tractability of robust formulations such as (AARC) is essentially driven by how the uncertain strength criterion G depends on $\boldsymbol{\zeta}$. Unfortunately, we are not aware of any general results. However, in most applications, such uncertain constraints can be written in the following form:

$$g(\boldsymbol{\sigma} + \boldsymbol{\Sigma}\boldsymbol{\zeta}) \leq 1 - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{\zeta}, \quad \forall \boldsymbol{\zeta} \in \mathcal{U} \tag{9}$$

with $\boldsymbol{\sigma} \in \mathbb{R}^d$, $\boldsymbol{\Sigma} \in \mathbb{R}^{d\times m}$, d being the dimension of the stress space, $\boldsymbol{b} \in \mathbb{R}^m$ and g is a convex homogeneous function.

Exact reformulations of such a constraint are possible only if G or $\mathcal{U}$ is polyhedral. In the general case, one can benefit from the following safe approximation due to [9]: the robust constraint (9) can be safely approximated as follows:

$$g(\boldsymbol{\sigma}) + \|\boldsymbol{s}\|_* \leq 1 \tag{10}$$

where $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$ defined as:

$$\|\boldsymbol{z}\|_* = \sup_{\|\boldsymbol{x}\|\leq 1} \boldsymbol{z}^{\mathrm{T}}\boldsymbol{x} \tag{11}$$

and where for $j = 1, \ldots, m$:

$$s_j = \max\{g(\boldsymbol{\Sigma}_j) + b_j, g(-\boldsymbol{\Sigma}_j) - b_j\} \tag{12}$$

with $\boldsymbol{\Sigma}_j$ denoting the j-th column of $\boldsymbol{\Sigma}$.

3.2 Illustrative Application on a Robust Mohr-Coulomb Criterion

Let us consider the case of a Mohr-Coulomb strength criterion where the cohesion c and the friction angle ϕ are uncertain. A negative correlation is often encountered between both parameters, i.e. soils with low cohesion tend to exhibit higher friction angles than with higher cohesion. We denote by ρ the correlation coefficient between c and ϕ, with typical values ranging from -0.5 to -0.9 [27]. Let us therefore consider that $\boldsymbol{k} = (c, \phi)$ is given by:

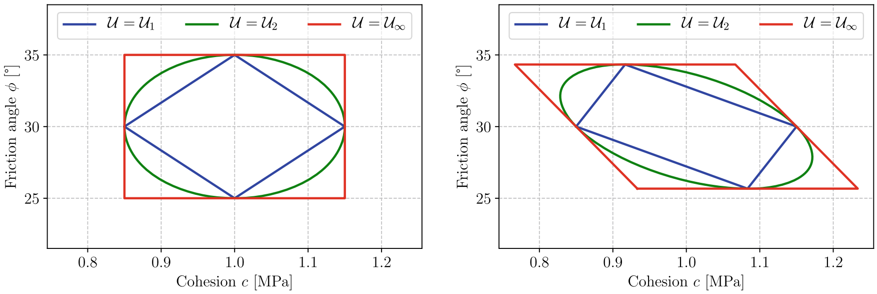
$$\boldsymbol{k}(\boldsymbol{\zeta}) = \boldsymbol{k}_0 + \boldsymbol{K}\boldsymbol{\zeta}, \quad \text{for } \boldsymbol{\zeta} \in \mathcal{U} \tag{13}$$

where $\boldsymbol{k}_0$ corresponds to the nominal values and where the "correlation" matrix $\boldsymbol{K}$ is such that:

$$\boldsymbol{K}\boldsymbol{K}^{\mathrm{T}} = \begin{bmatrix} \Delta c^2 & \rho\Delta c\Delta\phi \\ \rho\Delta c\Delta\phi & \Delta\phi^2 \end{bmatrix} \quad \text{i.e. } \boldsymbol{K} = \begin{bmatrix} \Delta c & 0 \\ \rho\Delta\phi & \Delta\phi\sqrt{1-\rho^2} \end{bmatrix} \tag{14}$$

where Δc, $\Delta\phi$ are the parameters typical variations and are assumed to be positive. Note that if such variations were taken as the standard deviations of the corresponding parameters, $\boldsymbol{K}\boldsymbol{K}^{\mathrm{T}}$ would be the corresponding covariance matrix.

Figure 2a illustrates the corresponding uncertainty sets obtained in the case $c = 1$ MPa, $\phi_0 = 30°$, $\Delta c = 150$ kPa, $\Delta\phi = 5°$, $\rho = 0$ and for various choices for the norm involved in the definition of $\mathcal{U}$, resulting in a corresponding L_1 (diamond shape), L_2 (elliptic shape) or L_∞ (rectangular shape) ball in physical space. Figure 2b shows the same uncertainty sets in the case of a negative correlation $\rho = -0.5$ which results in similar polyhedral or elliptic sets skewed along the negative diagonal which encodes the negative correlation coefficient. Let us point out that the previous choices for the uncertainty set result in simple convex set but more complex sets could also be considered, based for instance on available data regarding cohesion and friction angle pairs.



(a) $\rho = 0$ (no correlation) (b) $\rho = -0.5$ (negative correlation)

Fig. 2 Uncertainty sets of cohesion and friction angles for $c = 1$ MPa, $\phi_0 = 30°$, $\Delta c = 150$ kPa, $\Delta\phi = 5°$ for various sets $\mathcal{U}_p$ corresponding to a L_p unit ball

The robust counterpart of the Coulomb criterion therefore reads:

$$\sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\phi(\boldsymbol{\zeta}) - 2c(\boldsymbol{\zeta})\cos\phi(\boldsymbol{\zeta}) \leq 0 \qquad \forall\boldsymbol{\zeta} \in \mathcal{U} \tag{15}$$

where σ_1 (resp. σ_3) is the maximum (resp. minimum) principal stress.

Assuming that the variations Δc, $\Delta\phi$ are small, linearization around $\boldsymbol{k}_0$ results in:

$$\begin{aligned}\sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)&(\sin\phi_0 + \cos(\phi_0)(K_{21}\zeta_1 + K_{22}\zeta_2)) \\ &- 2(c_0 + K_{11}\zeta_1)\cos\phi_0 \\ &+ 2c_0\sin\phi_0(K_{21}\zeta_1 + K_{22}\zeta_2) \leq 0 \qquad \forall\boldsymbol{\zeta} \in \mathcal{U}\end{aligned} \tag{16}$$

with K_{ij} being the components of $\boldsymbol{K}$ defined in (14).

This yields the following robust counterpart:

$$\sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\phi_0 - 2c_0\cos\phi_0 + \|\boldsymbol{s}\|_* \leq 0 \tag{17}$$

where:

$$\boldsymbol{s} = \begin{pmatrix} |\left((\sigma_1 + \sigma_3)\cos\phi_0 + 2c_0\sin\phi_0\right)\rho\Delta\phi - 2\Delta c\cos\phi_0| \\ |(\sigma_1 + \sigma_3)\cos(\phi_0) + 2c_0\sin\phi_0|\sqrt{1-\rho^2}\Delta\phi \end{pmatrix} \tag{18}$$

Let us now investigate the simple case of no cross-correlation $\rho = 0$ with $\mathcal{U} = \{(\zeta_1, \zeta_2) \text{ s.t. } \|\boldsymbol{\zeta}\|_\infty \leq 1\}$. The previous expression reduces to:

$$\boldsymbol{s} = \begin{pmatrix} 2\Delta c\cos\phi_0 \\ \left((\sigma_1 + \sigma_3)\cos(\phi_0) + 2c_0\sin\phi_0\right)\Delta\phi \end{pmatrix} \tag{19}$$

$$\|\boldsymbol{s}\|_* = \|\boldsymbol{s}\|_1 = 2\Delta c\cos\phi_0 + |(\sigma_1 + \sigma_3)\cos(\phi_0) + 2c_0\sin\phi_0|\,\Delta\phi \tag{20}$$

so that the robust Mohr-Coulomb criterion (17) reduces to:

$$\begin{aligned}&\sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\phi_0 \\ &+ |(\sigma_1 + \sigma_3)\cos(\phi_0) + 2c_0 \sin\phi_0|\, \Delta\phi \leq 2(c_0 - \Delta c)\cos\phi_0\end{aligned} \tag{21}$$

which can be further expressed as follows:

$$\begin{cases}\sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)(\sin\phi_0 + \cos(\phi_0)\Delta\phi) \leq 2c_{\min}\cos\phi_0 - 2c_0 \sin\phi_0 \Delta\phi \\ \sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)(\sin\phi_0 - \cos(\phi_0)\Delta\phi) \leq 2c_{\min}\cos\phi_0 + 2c_0 \sin\phi_0 \Delta\phi\end{cases} \tag{22}$$

where $c_{\min} = c_0 - \Delta c$ is the worst-case cohesion. Introducing $\phi_{\min} = \phi_0 - \Delta\phi$ the worst-case friction angle and $\phi_{\max} = \phi_0 + \Delta\phi$ the best-case friction angle and using the fact that $\sin(\phi_{\max/\min}) \approx \sin\phi_0 \pm \cos(\phi_0)\Delta\phi$ and $\cos(\phi_{\max/\min}) \approx \cos\phi_0 \mp \sin(\phi_0)\Delta\phi$, the previous criterion is, in fact, a first-order approximation (in terms of Δc, $\Delta\phi$) to the following multi-surface criterion:

$$\begin{cases}\sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\phi_{\max} \leq 2c_{\min}\cos(\phi_{\max}) \\ \sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\phi_{\min} \leq 2c_{\min}\cos(\phi_{\min})\end{cases} \tag{23}$$

i.e. the obtained robust counterpart, for this specific case, (approximately) corresponds to the intersection of two Coulomb criteria with the worst-case cohesion and either the best or the worst-case friction angle. An illustration of such a result is given in Fig. 3. The yield surface corresponding to random realizations of $c(\boldsymbol{\zeta})$ and $\phi(\boldsymbol{\zeta})$ are also represented. One can indeed see that the obtained robust strength criterion forms a tight lower bound to the various realizations and is made of two sets of lines approximately characterized by the minimum and maximum friction angle $\phi_{\min}$ and $\phi_{\max}$.

4 Solving Robust Limit Analysis Problems

4.1 Strength Uncertainty with Static Formulation

As discussed before, for a limit analysis problem with uncertain strength conditions, we can replace the original uncertain strength criterion by its robust counterpart when using a static decision rule for the stress field. This approximation is obviously conservative and can provide reasonable estimates of the robust limit load only when the uncertainty if of small amplitude so that the optimal stress field does not heavily depend on the uncertainty realization, making static decision rules relevant.

In this case, the resulting robust limit analysis problem is equivalent to a classical deterministic limit analysis problem in which the nominal strength criterion has been

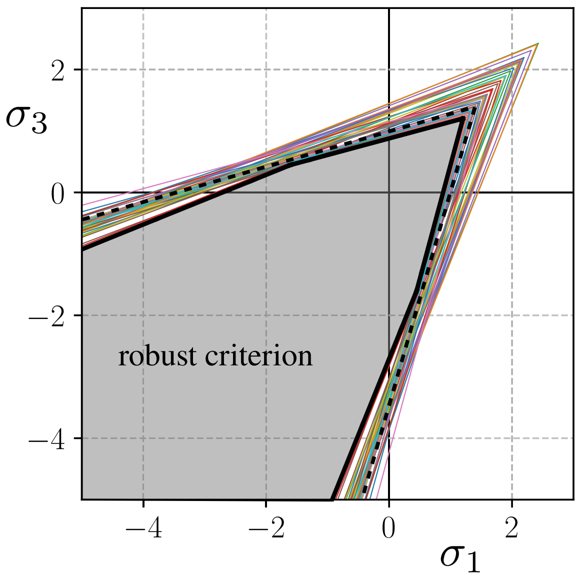


Fig. 3 Robust and uncertain Mohr-Coulomb criterion: $c_0 = 1$ MPa, $\phi_0 = 30°$, $\Delta c = 150$ kPa, $\Delta\phi = 5°$. Black dashed lines denote the nominal surface, thin coloured lines denote random realizations of the uncertain criterion. The robust domain is represented in gray and delimited by thick black lines

replaced by a smaller robust strength criterion. For a concrete implementation, the latter has to be formulated using tractable convex constraints.

As an illustration, we consider a slope stability problem for a cohesive-frictional soil with uncertain values for the cohesion and friction angle ($c = 1 \pm 0.1$ MPa and $\phi = (30 \pm 10)°$) for a pseudo-static earthquake loading $\boldsymbol{f} = (0.2g, -g)$. The corresponding load factor is interpreted here as the slope safety factor which should be larger than 1 to guarantee stability. The problem numerical resolution relies on a general-purpose domain-specific language (DSL), called `fenics_optim`, dedicated to automating the formulation and resolution of convex variational problems in a finite-element setting. The package is implemented as an add-on to the FEniCS Python interface and enables to easily formulate convex optimization problems using only a few lines of code and to discretize them in a very simple manner using various finite-element interpolation spaces. Their numerical resolution is performed efficiently using Mosek as the underlying conic programming solver [20]. More details regarding the package can be found in [11, 12] for its specific usage in the context of limit analysis.

Figure 4 represents the empirical distribution of the slope safety factor obtained for 200 random realizations of the material parameters. The nominal safety factor is slightly larger than 3 whereas the robust estimate is slightly less than 2 and indeed corresponds to the lower bound of the empirical distribution. This figure illustrates the advantage of using a robust formulation since, instead of running 200 LA computations, one is able to obtain an accurate estimate of the left part of the empirical distribution tail with a single computation. In this present case, only two uncertain parameters have been considered but the approach can be extended to a larger number of parameters. A typical example would be the modeling of soil spatial variability

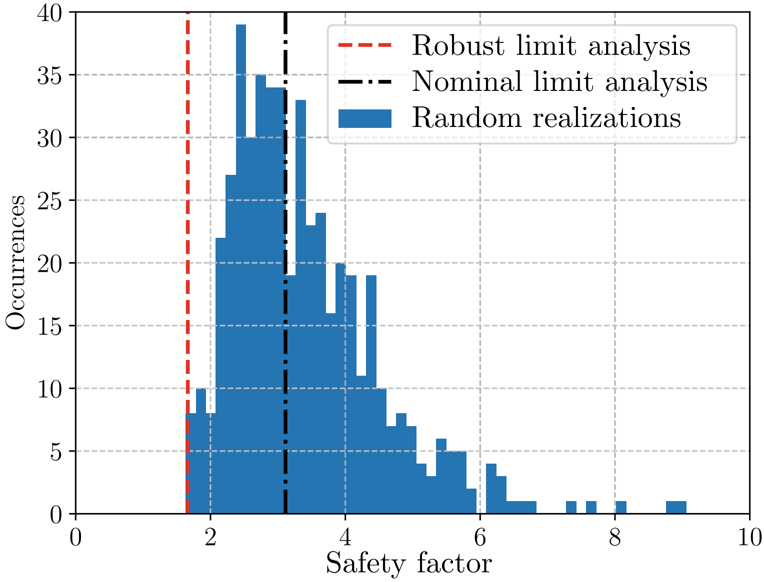


Fig. 4 Empirical distribution of the slope stability safety factor. The vertical black and red lines correspond to a single deterministic limit analysis with either nominal strength properties or using the corresponding robust strength condition

using random fields for instance. Besides, it can also be noted that the obtained estimate is not too conservative since a non-negligible number of uncertainty realizations are associated with a safety factor close to this robust estimate. Finally, it has to be pointed out that the variability on the friction angle induces a large variability on the obtained safety factor, explaining the difference between a nominal factor of 3 and a robust estimate around 2. This observation is further confirmed by the shape of the collapse mechanisms represented in Fig. 5. In the robust case, the collapse mechanism involves a much larger volume of soil than the nominal case since the most critical scenario corresponds to a smaller friction angle. Estimating the amount of soil mass mobilized during slope failure is an important point when assessing the

(a) Nominal collapse mechanism (b) Robust collapse mechanism

Fig. 5 Collapse mechanism and concentrated dissipation in slip lines for the nominal and robust case stability of a slope and its potential of damage in case of failure. Again, one can see that robust limit analysis computations can also be used to obtain a worst-case estimate of such a mobilized soil mass when accounting for uncertainty on the soil material parameters.

4.2 Loading Uncertainties

Similarly to [16, 19], we assume here that the fixed distributed and surface loadings $\boldsymbol{f}^{\mathrm{f}}, \boldsymbol{t}^{\mathrm{f}}$ are uncertain and vary, around a nominal value, inside a convex set. In particular, we consider that the reference loadings $\boldsymbol{f}^{\mathrm{r}}, \boldsymbol{t}^{\mathrm{r}}$ are deterministic. Assuming them to be uncertain adds another layer of difficulty due to the fact that the loading direction along which one has to optimize depends on the uncertainty realization. This specific case will be left for a future contribution.

Without loss of generality, we characterize the uncertain variation of the fixed loadings as follows:

$$\boldsymbol{f}^{\mathrm{f}}(\boldsymbol{\zeta}) = \boldsymbol{f}_0^{\mathrm{f}} + \sum_{j=1}^{m} \boldsymbol{f}_j^{\mathrm{f}} \zeta_j = \boldsymbol{f}_0^{\mathrm{f}} + \boldsymbol{F}^{\mathrm{f}}\boldsymbol{\zeta} \tag{24a}$$

$$\boldsymbol{t}^{\mathrm{f}}(\boldsymbol{\zeta}) = \boldsymbol{t}_0^{\mathrm{f}} + \sum_{j=1}^{m} \boldsymbol{t}_j^{\mathrm{f}} \zeta_j = \boldsymbol{t}_0^{\mathrm{f}} + \boldsymbol{T}^{\mathrm{f}}\boldsymbol{\zeta} \tag{24b}$$

where we introduced the matrices $\boldsymbol{F}^{\mathrm{f}} = [(\boldsymbol{f}_j^{\mathrm{f}})_{j=1,\dots,m}]$ and $\boldsymbol{T}^{\mathrm{f}} = [(\boldsymbol{t}_j^{\mathrm{f}})_{j=1,\dots,m}]$ and where $\boldsymbol{\zeta} \in \mathcal{U}$ with $\mathcal{U}$ a given convex uncertainty set. The corresponding uncertain limit analysis problem therefore reads:

$$\begin{aligned} \lambda^+(\boldsymbol{\zeta}) = \max_{\lambda,\boldsymbol{\sigma}} \ & \lambda \\ \text{s.t.} \ & \div\boldsymbol{\sigma} + \lambda\boldsymbol{f}^{\mathrm{r}} + \boldsymbol{f}_0^{\mathrm{f}} + \boldsymbol{F}^{\mathrm{f}}\boldsymbol{\zeta} = 0 \\ & \boldsymbol{\sigma}\cdot\boldsymbol{n} = \lambda\boldsymbol{t}^{\mathrm{r}} + \boldsymbol{t}_0^{\mathrm{f}} + \boldsymbol{T}^{\mathrm{f}}\boldsymbol{\zeta} \\ & \boldsymbol{\sigma} \in G \end{aligned} \tag{25}$$

4.3 Robust Counterpart

Clearly, for this load uncertainty case, the use of static decision rules is doomed to fail since one cannot expect finding, except in very specific cases, a single stress field which is statically admissible with any realization of the uncertain loading (24). One must therefore resort to an adjustable robust optimization which, similarly to (2), reads:

$$\begin{aligned}\lambda_{\text{ARC}} = \max_{\bar{\lambda}}\ & \bar{\lambda} \\ \text{s.t. } & \forall \boldsymbol{\zeta} \in \mathcal{U}, \exists \boldsymbol{\sigma}, \lambda \text{ s.t. } \begin{array}{l} \div \boldsymbol{\sigma} + \lambda \boldsymbol{f}^{\text{r}} + \boldsymbol{f}_0^{\text{f}} + \boldsymbol{F}^{\text{f}}\boldsymbol{\zeta} = 0 \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \lambda \boldsymbol{t}^{\text{r}} + \boldsymbol{t}_0^{\text{f}} + \boldsymbol{T}^{\text{f}}\boldsymbol{\zeta} \\ \boldsymbol{\sigma} \in G \\ \bar{\lambda} \leq \lambda \end{array}\end{aligned} \tag{26}$$

Again, in order to obtain a safe and tractable approximation to the above robust formulation, we resort to the use of the affine decision rules (6) and obtain the following AARC:

$$\begin{aligned}\lambda_{\text{AARC}} = \max_{\boldsymbol{\sigma}_i, \lambda_i} \min_{\boldsymbol{\zeta} \in \mathcal{U}}\ & \lambda_0 + \sum_{j=1}^{m} \lambda_j \zeta_j \\ \text{s.t. } & \div \left(\boldsymbol{\sigma}_0 + \sum_{j=1}^{m} \boldsymbol{\sigma}_j \zeta_j \right) + \left(\lambda_0 + \sum_{j=1}^{m} \lambda_j \zeta_j \right) \boldsymbol{f}^{\text{r}} + \boldsymbol{f}_0^{\text{f}} + \boldsymbol{F}^{\text{f}}\boldsymbol{\zeta} = 0 \\ & \left(\boldsymbol{\sigma}_0 + \sum_{j=1}^{m} \boldsymbol{\sigma}_j \zeta_j \right) \cdot \boldsymbol{n} = \left(\lambda_0 + \sum_{j=1}^{m} \lambda_j \zeta_j \right) \boldsymbol{t}^{\text{r}} + \boldsymbol{t}_0^{\text{f}} + \boldsymbol{T}^{\text{f}}\boldsymbol{\zeta} \\ & \left(\boldsymbol{\sigma}_0 + \sum_{j=1}^{m} \boldsymbol{\sigma}_j \zeta_j \right) \in G\end{aligned} \tag{27}$$

which can be further formulated as follows:

$$\begin{aligned}\lambda_{\text{AARC}} = \max_{\bar{\lambda}, \boldsymbol{\sigma}_i, \lambda_i}\ & \bar{\lambda} \\ \text{s.t. } & \div(\boldsymbol{\sigma}_i) + \lambda_i \boldsymbol{f}^{\text{r}} + \boldsymbol{f}_i^{\text{f}} = 0 \quad \forall i = 0, \ldots, m \\ & \boldsymbol{\sigma}_i \cdot \boldsymbol{n} = \lambda_i \boldsymbol{t}^{\text{r}} + \boldsymbol{t}_i^{\text{f}} \quad \forall i = 0, \ldots, m \\ & \left(\boldsymbol{\sigma}_0 + \sum_{j=1}^{m} \boldsymbol{\sigma}_j \zeta_j \right) \in G \quad \forall \boldsymbol{\zeta} \in \mathcal{U} \\ & \bar{\lambda} \leq \lambda_0 + \sum_{j=1}^{m} \lambda_j \zeta_j \quad \forall \boldsymbol{\zeta} \in \mathcal{U}\end{aligned} \tag{28}$$

Clearly, (28) bears striking similarities with (AARC) in the sense that we look for $1 + m$ stress fields statically admissible with a given loading (here we have an additional fixed loading for each $j = 1, \ldots, m$ compared to (AARC)). In particular, uncertainty has been removed from the equilibrium equations whereas only the last two constraints are robust ones which must be reformulated. In particular, the robust strength constraint can be reformulated, either exactly or approximately, using the results of Sect. 3. Finally, the last constraint can be reformulated as follows using the dual norm $\|\cdot\|_*$ to the norm involved in the definition of the uncertainty set $\mathcal{U}$. Indeed, introducing the vector $\boldsymbol{\Lambda} = (\lambda_j)_{j=1,\ldots,m}$, we can write:

$$\begin{aligned} & \bar{\lambda} \leq \lambda_0 + \boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\zeta} \quad \forall \boldsymbol{\zeta} \in \mathcal{U} \\ \Leftrightarrow \quad & \bar{\lambda} + \max_{\boldsymbol{\zeta}\in\mathcal{U}} \{-\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\zeta}\} \leq \lambda_0 \\ \Leftrightarrow \quad & \bar{\lambda} + \|-\boldsymbol{\Lambda}\|_* \leq \lambda_0 \end{aligned} \tag{29}$$

which results in a tractable convex constraint for classical uncertainty sets.

In conclusion, we see that the robust reformulation of (28) is close to a classical limit analysis problem except that the number of stress fields and load factor is now $1 + m$ and that the strength criterion will couple all stress variables in a single constraint which would have been exactly or approximately reformulated to guarantee the robust constraint $\boldsymbol{\sigma}_0 + \sum_{j=1}^{m} \boldsymbol{\sigma}_j \zeta_j \in G, \ \forall \boldsymbol{\zeta} \in \mathcal{U}$. As a result, the resulting robust problem will still be convex and representable using conic constraints. It will however be much larger in size than a deterministic problem.

5 Conclusions

In this work, we have proposed an extension of limit analysis theory to an uncertain setting using the robust optimization (RO) framework. Since limit analysis problems can be formulated as convex optimization programs, we can naturally apply robust optimization concepts when considering uncertain data. We covered two different sources of uncertainty, namely strength and loading uncertainty.

An important aspect of RO is related to the use of static or adjustable optimization variables. In the present LA case, it amounts to deciding whether we consider the stress field and load multiplier that we optimize for to be independent or dependent on the uncertain parameters. The main feature of RO is to propose tractable reformulations of uncertain constraints as standard deterministic constraints, possibly involving a much larger number of variables. Various results have been obtained for the two cases of static and adjustable formulations.

First, the use of static variables results in the *static robust counterpart* (RC) which deserves the following comments:

- (RC) is a standard deterministic LA problem where the uncertain strength criterion is replaced with a safe estimate called the *robust strength domain* G_{RC}.
- The robust strength domain is the smallest possible strength domain corresponding to all uncertainty realizations.
- Obtaining an explicit expression for the robust domain depends on how constraints depend on the uncertain parameters.
- Tractable approximations of the robust domain have been provided and illustrated on the case of a Mohr-Coulomb example.
- The resulting LA problem can be solved using standard tools and the resulting load estimate is a conservative safe approximation for all realization.

Clearly, this is a very conservative approach. In particular, finding such a stress field is not always possible. Our experience suggests that static formulations can be used only when considering strength uncertainty and in the case where this uncertainty is of small amplitude. Intuitively, this corresponds to the fact that the collapse stress field is only mildly perturbed by the realization of the uncertainty.

Second, in the general case where adjustable formulations are needed, simple decision rules must be chosen for the robust problem to be tractable. In particular, the case of loading uncertainty can only be tackled using adjustable formulations. More precisely:

- *Affine decision rules* assume an affine dependence of the load factor and stress field with respect to the uncertain parameters.
- Robust strength constraints take the form (9) which can be reformulated either exactly or approximately.
- The corresponding affinely adjustable problem can be reformulated to yield the deterministic optimization problem (AARC).
- The latter involves a much larger number of optimization variables compared to the nominal limit analysis problem. This number depends on the dimension of the uncertainty space.

Further research will focus on the numerical implementation of the proposed formulations in order to assess their efficiency on more involved examples. In this respect, specific strategies should probably be investigated in order to reduce the computational cost of the corresponding large-scale optimization problems, especially when considering AARC formulations. Analyzing such more advanced examples would therefore shed light on the necessity, or not, of considering more complex decision rules than affine rules such as piecewise-linear or nonlinear decision rules.

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