EXACT QUANTIZATION OF MULTISTAGE STOCHASTIC LINEAR PROBLEMS

MAËL FORCIER *, STÉPHANE GAUBERT †, AND VINCENT LECLÈRE ‡

Abstract. We show that the Multistage Stochastic Linear Problem (MSLP) with an arbitrary cost distribution is equivalent to a MSLP on a finite scenario tree. We establish this exact quantization result by analyzing the polyhedral structure of MSLPs. In particular, we show that the expected cost-to-go functions are polyhedral and affine on the cells of a chamber complex, which is independent of the cost distribution. This leads to new complexity results, showing that MSLP becomes polynomial when certain parameters are fixed.

1. Introduction.

Stochastic programming is a powerful modeling paradigm for optimization under uncertainty that has found many applications in energy, logistics or finance (see *e.g.,* [49]). Multistage Stochastic Linear Problems (MSLP) constitute an important class of stochastic programs. They have been thoroughly studied, see *e.g.,* [5, 42]. One reason for this interest is the availability of efficient linear solvers and the use of dedicated algorithms leveraging the special structure of linear stochastic programs ([54, 4]).

In this paper, we show that every MSLP with general cost distribution is equivalent to an MSLP with finite distribution. This leads to explicit representations of their value functions and to new complexity results.

1.1. Multistage stochastic linear programming.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Given a sequence of independent random variables $\boldsymbol{c}_t \in L^1(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^{n_t})$ and $\boldsymbol{\xi}_t = (\boldsymbol{A}_t, \boldsymbol{B}_t, \boldsymbol{b}_t)$, with $t \in [T] := \{1, \dots, T\}$, we consider the MSLP given by

$$
\begin{aligned}
\min_{(\boldsymbol{x}_t)_{t\in[T]}} \quad & c_1^\top x_1 + \mathbb{E}\big[\sum_{t=2}^{T} \boldsymbol{c}_t^\top \boldsymbol{x}_t\big] \\
\text{s.t.} \quad & A_1 x_1 \leqslant b_1, \\
& \boldsymbol{A}_t \boldsymbol{x}_t + \boldsymbol{B}_t \boldsymbol{x}_{t-1} \leqslant \boldsymbol{b}_t \quad \text{a.s.} \quad && \forall t \in \{2, \dots, T\}, \\
& \boldsymbol{x}_t \in L_\infty(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^{n_t}) && \forall t \in \{2, \dots, T\}, \\
& \boldsymbol{x}_t \preccurlyeq \mathcal{F}_t && \forall t \in \{2, \dots, T\},
\end{aligned}
\tag{1.1}
$$

where $\boldsymbol{x}_1 \equiv x_1$, $\boldsymbol{A}_1 \equiv A_1$ and $\boldsymbol{b}_1 \equiv b_1$ are deterministic and $\mathcal{F}_t$ is the σ-algebra generated by $(\boldsymbol{c}_2, \boldsymbol{\xi}_2, \dots, \boldsymbol{c}_t, \boldsymbol{\xi}_t)$. The last constraint, known as nonanticipativity, means that $\boldsymbol{x}_t$ is measurable with respect to $\mathcal{F}_t$.

Most results for MSLP with continuous distributions rely on discretizing the distributions. The Sample Average Approximation (SAA) method (see *e.g.,* [49, Chap. 5]) samples the costs and constraints. It relies on probabilistic results based on a uniform law of large number to give statistical guarantees. Obtaining a good approximation requires a large number of scenarios. In order to alleviate the computations, we can use scenario reduction techniques (see [14, 27]). Latin Hypercube Sampling (LHS) and variance reduction methods are also used to produce scenarios. Finally, one generates heuristically "good" scenarios, representing the underlying distribution (see [28]). Alternatively, we can leverage the structure of the problem to produce

*CERMICS, École des Ponts, mael.forcier@enpc.fr
†INRIA, CMAP, École polytechnique, IP Paris, CNRS, stephane.gaubert@inria.fr
‡CERMICS, École des Ponts, vincent.leclere@enpc.fr

finite scenario trees (see [30, 37, 16]) that yields bounds for the value of the true optimization problem. In each of these approaches, one solves an approximate version of the stochastic program, with or without statistical guarantee.

With the independence assumption, Problem (1.1) is often tackled through Dynamic Programming approaches. One well-developed approach is the Stochastic Dual Dynamic Programming algorithm (SDDP) [40, 48], and its brethren, largely used in energy applications. Until the recent work [18], leveraging the tools developed here, these algorithms required finitely supported distribution, often obtained through SAA.

1.2. The exact quantization problem. Here, we aim at solving exactly the original problem, by finding an equivalent formulation with discrete distributions. This notion of equivalent formulation is best understood through the dynamic programming approach of MSLP. We define the *cost-to-go* function V_t inductively as follows. We set $V_{T+1} \equiv 0$ and for all $t \in \{2, \dots, T\}$:

$$\begin{aligned} V_t(x_{t-1}) &:= \mathbb{E}\big[\hat{V}_t(x_{t-1}, \boldsymbol{c}_t, \boldsymbol{\xi}_t)\big], \\ \hat{V}_t(x_{t-1}, c_t, \xi_t) &:= \min_{x_t \in \mathbb{R}^{n_t}} \quad c_t^\top x_t + V_{t+1}(x_t) \\ &\qquad\quad \text{s.t.} \quad A_t x_t + B_t x_{t-1} \leqslant b_t. \end{aligned} \tag{1.2}$$

where $x_{t-1} \in \mathbb{R}^{n_{t-1}}$, $c_t \in \mathbb{R}^{n_t}$ and $\xi_t := (A_t, B_t, b_t) \in \mathbb{R}^{\ell_t \times n_t} \times \mathbb{R}^{\ell_t \times n_{t-1}} \times \mathbb{R}^{\ell_t} = \Xi_t$.

We choose to distinguish the random cost $\boldsymbol{c}_t$ from the noise $\boldsymbol{\xi}_t$ affecting the constraints. Indeed our results require $\boldsymbol{\xi}_t$ to be finitely supported (see **??** and Example 1) while $\boldsymbol{c}_t$ can have a continuous distribution. This separation does not preclude correlation between $\boldsymbol{c}_t$ and $\boldsymbol{\xi}_t$. However, we require $\{(\boldsymbol{c}_t, \boldsymbol{\xi}_t)\}_{t\in[T]}$ to be a sequence of independent random variables to leverage Dynamic Programming, even though some results can be extended to dependent $(\boldsymbol{\xi}_t)_{t\in[T]}$.

We say that a MSLP (with stagewise independence) admits a *local exact quantization* at time t at x_{t-1} if there exists a finitely supported $(\check{\boldsymbol{c}}_t, \check{\boldsymbol{\xi}}_t)_{t\in[T]}$ that yields the same expected cost-to-go functions *i.e.*, such that

$$V_t(x_{t-1}) = \mathbb{E}\big[\hat{V}_t(x_{t-1}, \boldsymbol{c}_t, \boldsymbol{\xi}_t)\big] = \mathbb{E}\big[\hat{V}_t(x_{t-1}, \check{\boldsymbol{c}}_t, \check{\boldsymbol{\xi}}_t)\big].$$

A quantization is *uniform* if it is locally exact at all $x_{t-1} \in \mathbb{R}^{n_t}$, and all $t \in [T]$.

COROLLARY 1.1. *If there exists a uniform exact quantization for Problem* (1.1)*, then the expected cost-to-go functions V_t are polyhedral.*

Proof. It is well known (see *e.g.,* , [49, prop 2.15]) that a finitely supported MSLP admits polyhedral expected cost-to-go functions. ∎

EXAMPLE 1 (No uniform exact for stochastic constraints). *Here,* $\boldsymbol{u}$ *denotes a uniform random variable on* $[0, 1]$*. We consider two simple example with stochastic* $\boldsymbol{B}$ *and* $\boldsymbol{b}$ *respectively.*

$$V^1(x) = \mathbb{E}\begin{bmatrix} \min\limits_{y\in\mathbb{R}} & y \\ s.t. & \boldsymbol{u}x \leqslant y \\ & 1 \leqslant y \end{bmatrix} = \mathbb{E}\big[\max(\boldsymbol{u}x, 1)\big] = \begin{cases} 1 & \text{if } x \leqslant 1 \\ \frac{x}{2} + \frac{1}{2x} & \text{if } x \geqslant 1 \end{cases}.$$

$$V^2(x) = \mathbb{E}\begin{bmatrix} \min\limits_{y\in\mathbb{R}} & y \\ s.t. & \boldsymbol{u} \leqslant y \\ & x \leqslant y \end{bmatrix} = \mathbb{E}\big[\max(x, \boldsymbol{u})\big] = \begin{cases} \frac{1}{2} & \textit{if } x \leqslant 0 \\ \frac{x^2+1}{2} & \textit{if } x \in [0,1] \\ x & \textit{if } x \geqslant 1 \end{cases}.$$

As both cost-to-go functions are not polyhedral, we cannot hope to find uniform exact quantizations in these cases.

1.3. Contribution. We develop a geometric approach, which enlightens the polyhedral structure of MSLP. We first establish exact quantization results in the 2-stage case showing that there exists an optimal recourse affine on each cell of a polyhedral complex which is precisely the chamber complex [3, 44], a fundamental object in combinatorial geometry. A chamber complex is defined as the common refinement of the projections of faces of a polyhedron. In particular, Theorem 3.2 provides an explicit exact quantization, in which the quantized probabilities and costs are attached to the cones of a polyhedral fan $\mathcal{N}$ (we refer the reader to [13, 58, 25, 20] for background on polyhedral complexes and fans). On each cone $N \in \mathcal{N}$, we replace the distribution of $\boldsymbol{c}\mathbb{1}_{\operatorname{ri} N}$, where $\operatorname{ri} N$ stand for the relative interior of N, by a Dirac distribution concentrated on the expected value $\check{c}_N = \mathbb{E}\big[\boldsymbol{c}|\boldsymbol{c} \in \operatorname{ri} N\big]$, and an associated weight $\check{p}_N = \mathbb{P}\big[\boldsymbol{c} \in \operatorname{ri} N\big]$. Further, $\mathcal{N}$ is *universal* in the sense that it does not depend on the distribution of $\boldsymbol{c}$.

In order to extend this result to the multistage case we establish in Lemma 4.1 a Dynamic Programming type equation in the space of polyhedral complexes. Then we show an exact quantization result in Theorem 4.6.

We apply this polyhedral approach to obtain polynomial time complexity results considering both the exact computation problem and the approximation problem, when certain parameters are fixed. For distributions that are uniform on polytopes or exponential, we show the MSLP can be solved in a time that is polynomial provided that the horizon T and the dimensions $n_2, \dots, n_T$ of the successive recourses are fixed. The proof relies on the theory of linear programming with oracles [24] as well as on upper bound theorems of McMullen [39] and Stanley [52] concerning the number of vertices and the size of a triangulation of a polyhedron. We obtain a similar result for the approximation problem. This is more widely applicable since the distribution cost can now be essentially arbitrary; we only assume that it is given implicitly through an appropriate oracle (see Definition 5.10) – this applies in particular to any distribution with a smooth density with respect to Lebesgue measure.

In summary, our main contributions, shedding light on the geometry of polyhedral stochastic programming problems, are the following:

1. MSLP with arbitrary cost distribution and finitely supported constraints admit a *uniform* exact quantization result, *i.e.*, are equivalent to MSLP with discrete cost distribution;
2. The expected cost-to-go functions of such MSLP are polyhedral and affine on the cells of a universal polyhedral complex (*i.e.*, independent of the cost distribution) which is precisely the chamber complex;
3. In the 2-stage case, the expected cost-to-go function is characterized in terms of a weighted extension of the fiber polytope;
4. We give polynomial time complexity results for 2SLP and MSLP, in exact and approximate models of computations, when certain parameters are fixed.

1.4. Comparison with related work. The pioneering work of Walkup and Wets [56] developed a combinatorial approach of deterministic parametric linear programming. Higher notions of polyhedral geometry, such as secondary fan and fiber polytopes, were subsequently introduced, with motivations from outside of optimization, by Gelfand, Kapranov and Zelevinsky [22] and by Billera and Sturmfels [3]. Thomas and Sturmfels [53] and later De Loera, Rambau and Santos [13] established important links between these concepts and (parametric) linear optimization. Fiber polytopes are still of considerable interest. In particular, Black, De Loera, Lütjeharms, and Sanyal applied recently a special class of fiber polytopes, the *monotone path polytopes*, in which the projection keeps track of the level-set value of the cost function, in order to classify simplex iterations [7], see also [6]. Moreover, a generalization of fiber polytopes to the non-polyhedral case, called "fiber convex bodies", has been recently considered [38]. Here, our contribution shows how polyhedral notions explain the quantization problem. Further, we consider general cost distributions in many of the statements, and in particular, we extend the notion of fiber polytope by considering non-uniform measures, which is needed in applications to stochastic optimization.

More precisely, the basis decomposition theorem of Walkup and Wets describes how the value of a linear program in standard form varies with respect to the cost and to the right-hand side of the constraints. In the 2-stage case, we can see the collection of rows of A as a vector configuration, and the right-hand side of the recourse problem $b - Bx$ as a height function which determines a regular subdivision of this configuration. The space of regular subdivision is represented by the so called secondary fan [13]. We may apply this theorem to the dual problem of the recourse problem to deduce that the expected cost-to-go function is affine on each cell of an affine section of the secondary fan. This affine section can be shown to coincide with the chamber complex used here. However, the basis decomposition theorem cannot be applied to the extensive form of a *multistage* problem. In particular nonanticipativity constraints cannot be tackled in this way. Thus, we choose to develop an approach through chamber complexes as it is more direct, allowing us to obtain also a result in the multistage case.

The complexity of stochastic programming has been extensively studied. Dyer and Stougie [15] proved that 2-stage stochastic programming with discrete distribution is $\sharp P$-hard, by reducing to it the problem of graph reliability. Hanasusanto, Kuhn and Wiesemann [26] showed that solving, with a sufficiently high accuracy, the 2-stage linear programming (2SLP) with continuous distribution is also $\sharp P$-hard, exploiting the $\sharp P$-completeness of the computation of the volume of knapsack polytopes and order polytopes. Shapiro and Nemirovski showed in [50] that 2SLP (and MSLP with fixed horizon) can be approximated, with high probability and up to precision ε, by the SAA method with a number of scenario polynomial in $1/\varepsilon$. Furthermore, [51] showed that 2SLP (also true for first-stage integer decision) can be solved, with high probability, in a pseudo-polynomial time, *i.e.*, polynomial in $1/\varepsilon$ and in the input size. In contrast, our approach shows that 2SLP and MSLP can be solved in polynomial time in $\log(1/\varepsilon)$ when certain parameters are fixed. Thus, a high accuracy is accessible, but only for a restricted class of instances. This should also be compared with results of Lan [31] and Zhang and Sun [57], who independently analyzed the complexity of SDDP. It follows from their results that finitely supported MSLP can be solved approximately in pseudo-polynomial time in the error approximation ε when all the dimensions and the horizon are fixed. In particular, the complexity of these SDDP methods is polynomially bounded in $1/\varepsilon$. In contrast, our approach shows that MSLP can be solved approximately in polynomial time in $\log(1/\varepsilon)$, when T, $n_2, \dots, n_T$ are fixed. In particular, the first state dimension is not fixed. Moreover, we obtain polynomial complexity bounds in the exact (Turing) model of computation for appropriate classes of distributions. Note that in the approach presented here, contrary to SDDP like methods, we do not rely on statistical sampling and the value functions are computed exactly in one pass only. However, the objective of SDDP is to obtain quickly an approximate solution whereas our approach computes exactly the epigraph of the expected cost-to-go function.

The complexity of multistage stochastic integer linear programs, with finitely supported distribution, have recently been studied in [29] based on results for two-stage integer programs compiled in [12, Chapter 4].

1.5. Structure of the paper. We recall, in Section 2, notions from the theory of polyhedra: *polyhedral complexes*, *normal fans* and *chamber complexes*. In Section 3 we establish the exact quantization result for 2SLP. In Section 4, we show that chamber complexes can be propagated through dynamic programming, leading to the exact quantization result for the MSLP. Finally, in Section 5, we draw the consequences of our results in terms of computational complexity.

1.6. Notation. As a general guideline ***bold*** letters denote random variables, normal scripts their realisation. Capital letters denote matrices or sets, calligraphic (*e.g.*, $\mathcal{N}$) denote collections of sets. The indicator function $\mathbb{I}_{\mathrm{P}}$ (resp. $\mathbb{1}_{\mathrm{P}}$) takes value 0 (resp. 1) if P is true and $+\infty$ (resp. 0) otherwise. We set $[k] := \{1, \dots, k\}$, and we denote by $\sharp E$ the cardinal of a set E. We denote by $\mathrm{Cone}(A) := A\mathbb{R}^n_+$ the conic hull of the columns of A. The inequality $x \leqslant y$ refers to the standard partial order, given by $\forall i, x_i \leqslant y_i$. We denote by $F \subset G$ if F is a subface of G. Further, $\mathrm{ri}(E)$ is the relative interior of the set E, *i.e.*, the greatest open set included in E for the topology of the smallest vector subspace containing E. Moreover, $\mathrm{dom}(f) = \{x \mid f(x) < +\infty\}$ is the domain of f, and $\mathrm{epi}(f) = \{(x, z) \mid f(x) \leqslant z\}$ the epigraph of f. Finally, $\sqcup$ denotes a disjoint union.

2. Polyhedral tools. Our proofs rely on the notions of normal fan and chamber complex of a polyhedron recalled here. These polyhedral objects reveal the geometrical structure of MSLP. Both the normal fan and the chamber complex are special polyhedral complexes.

2.1. Polyhedral complexes. *Polyhedral complexes* are collections of polyhedra satisfying some combinatorial and geometrical properties. In particular the relative interiors of the elements of a polyhedral complex (without the empty set) form a partition of their union. We refer to [13] for a complete introduction to polyhedral complexes and triangulations.

Definition 2.1 (Polyhedral complex). *A finite collection $\mathcal{C}$ of polyhedra is a* polyhedral complex *if it satisfies i) if $P \in \mathcal{C}$ and F is a non-empty*[1] *face of P then $F \in \mathcal{C}$ and ii) if P and Q are in $\mathcal{C}$, then $P \cap Q$ is a (possibly empty) face of P and Q. Elements of a polyhedral complex are called* cells. *We denote by* $\mathrm{supp}\,\mathcal{C} := \bigcup_{P \in \mathcal{C}} P$ *the support of a polyhedral complex. Further, if all the elements of $\mathcal{C}$ are polytopes (resp. cones, simplices, simplicial cones), we say that $\mathcal{C}$ is a* polytopal complex *(resp. a* fan*, a* simplicial complex*, a* simplicial fan*).*

[1] For some authors, a polyhedral complex must contain the empty set. We do not make this requirement.

We recall that a *simplex* of dimension d is the convex hull of $d+1$ affinely independent point and that a *simplicial cone* of dimension d is the conical hull of d linearly independent vectors.

Proposition 2.2. *For any polyhedral complex $\mathcal{C}$, the relative interiors of its elements (without the empty set) form a partition of its support:* $\operatorname{supp}(\mathcal{C}) = \bigsqcup_{P\in\mathcal{C}} \operatorname{ri}(P)$.

For example, the set of faces $\mathcal{F}(P)$ of a polyhedron P is a polyhedral complex.

Definition 2.3 (Refinements and triangulation). *Let $\mathcal{C}$ and $\mathcal{R}$ be two polyhedral complexes, we say that $\mathcal{R}$ is a* refinement *of $\mathcal{C}$, denoted $\mathcal{R} \preccurlyeq \mathcal{C}$, if* $\operatorname{supp}\mathcal{R} = \operatorname{supp}\mathcal{C}$ *and for every cell $R \in \mathcal{R}$ there exists a cell $C \in \mathcal{C}$ containing R: $R \subset C$.*

Note that $\preccurlyeq$ defines a partial order and the meet *associated with this order is given by the* common refinement *of two polyhedral complexes $\mathcal{C}$ and $\mathcal{C}'$ defined as the polyhedral complex of the intersections of cells of $\mathcal{C}$ and $\mathcal{C}'$*[2] *:*

$$\mathcal{C} \wedge \mathcal{C}' := \{R \cap R' \mid R \in \mathcal{C}, R' \in \mathcal{C}'\}.$$

A triangulation $\mathcal{T}$ of a polytope Q *is a refinement of $\mathcal{F}(Q)$ such that the cells of dimension 0 of $\mathcal{T}$ are the vertices of Q and $\mathcal{T}$ is a simplicial complex. A* triangulation $\mathcal{T}$ of a cone K *is a refinement of $\mathcal{F}(K)$ such that the cells of dimension 1 of $\mathcal{T}$ are the rays of K and $\mathcal{T}$ is a simplicial fan.*

2.2. Normal fan. The normal fan is the collection of the normal cones of all faces of a polyhedron. See [36] for a review of normal fan properties.

Recall that the *normal cone* of a convex set $C \subset \mathbb{R}^d$ at the point x is the set $N_C(x) := \{\alpha \in \mathbb{R}^d \mid \forall y \in C,\ \alpha^\top(y-x) \leqslant 0\}$. More generally, for a set $E \subset C$, $N_C(E) := \bigcap_{x\in E} N_C(x)$.

Figure 1: Two normally equivalent polytopes P and P' and their normal fan $\mathcal{N}(P) = \mathcal{N}(P')$. The green circle represents the singleton $\{0\}$ which is the normal cone $N_P(x)$ for every $x \in \operatorname{ri}(P)$.

Definition 2.4 (Normal fan). *The* normal fan[3] *of a convex set C is the collection of normal cones*

$$\mathcal{N}(C) := \{N_C(x) \mid x \in C\}.$$

We say that two convex sets C and C' are normally equivalent *if they have the same normal fan: $\mathcal{N}(C) = \mathcal{N}(C')$, see Figure 1.*

[2]We allow $\mathcal{C}$ and $\mathcal{C}'$ to have different supports. In that case, $\mathcal{C} \wedge \mathcal{C}'$ is well-defined but there is no common refinement. The support of $\mathcal{C} \wedge \mathcal{C}'$ is then equal to the intersection $\operatorname{supp}\mathcal{C} \cap \operatorname{supp}\mathcal{C}'$.

[3]Sometimes called *outer* normal cones and fan, as opposed to *inner* cones obtained either by inverting the inequality in the definition of the normal cone or by taking the opposite cones respect to the origin.

Recall that the *polar* of a convex set C is the set $C^\circ := \{\alpha \in \mathbb{R}^d \mid \forall x \in C,\ \alpha^\top x \leqslant 0\} = N_C(0)$ and the *recession cone* of a convex set C is given by $\mathrm{rc}(C) := \{r \in C \mid \forall \mu \in \mathbb{R}_+,\ \forall x \in C,\ x + \mu r \in C\}$. In particular, for a polyhedron, the recession cone and its polar are given by

$$\mathrm{rc}\left(\{x \mid Ax \leqslant b\}\right) = \{x \mid Ax \leqslant 0\} \qquad \mathrm{rc}\left(\{x \mid Ax \leqslant b\}\right)^\circ = \mathrm{Cone}(A^\top)\ . \tag{2.1}$$

Proposition 2.5 (Basic properties of normal fans (see *e.g.*, [36])).
If P is a polyhedron, the normal fan $\mathcal{N}(P)$ is a polyhedral complex. Further, the support of $\mathcal{N}(P)$ can be expressed as the polar of the recession cone of P, i.e.,

$$\mathrm{supp}\,\mathcal{N}(P) = \big(\mathrm{rc}(P)\big)^\circ. \tag{2.2}$$

2.3. Chamber complex. The affine regions of the cost-to-go function will correspond to cells of a chamber complex. Projections of polyhedra, fibers and chambers complexes are studied in [3, 44, 43].

Definition 2.6 (Chamber complex). *Let $P \subset \mathbb{R}^d$ be a polyhedron and π a linear projection defined on $\mathbb{R}^d$. For $x \in \pi(P)$ we define the* chamber of x for P along π *as*

$$\sigma_{P,\pi}(x) := \bigcap_{F \in \mathcal{F}(P)\ s.t.\ \ x \in \pi(F)} \pi(F).$$

The chamber complex $\mathcal{C}(P,\pi)$ *of P along π is defined as the (finite) collection of chambers,* i.e.,

$$\mathcal{C}(P,\pi) := \{\sigma_{P,\pi}(x) \mid x \in \pi(P)\}\,.$$

Further $\mathcal{C}(P,\pi)$ is a polyhedral complex such that $\mathrm{supp}\,\mathcal{C}(P,\pi) = \pi(P)$. In particular, $\big\{\,\mathrm{ri}(\sigma) \,|\, \sigma \in \mathcal{C}(P,\pi)\big\}$ is a partition of $\pi(P)$.
More generally, the chamber complex of a polyhedral complex $\mathcal{P}$ is

$$\mathcal{C}(\mathcal{P},\pi) := \{\sigma_{\mathcal{P},\pi}(x) \mid x \in \pi\big(\mathrm{supp}(\mathcal{P})\big)\}\,.$$

with $\sigma_{\mathcal{P},\pi}(x) := \bigcap_{F \in \mathcal{P}\ s.t.\ \ x \in \pi(F)} \pi(F).$

Lemma 2.7 (Chamber complex monotonicity with respect to refinement order). *Let $\mathcal{R} \preccurlyeq \mathcal{S}$ be polyhedral complexes of $\mathbb{R}^d$ and a projection π. Then, $\mathcal{C}(\mathcal{R},\pi) \preccurlyeq \mathcal{C}(\mathcal{S},\pi)$.*

Proof. For any $R \in \mathcal{R}$, there exists $S_R \in \mathcal{S}$ such that $R \subset S_R$. Let $x \in \mathrm{supp}\,\mathcal{C}(\mathcal{R},\pi) = \pi(\mathrm{supp}\,\mathcal{R}) = \pi(\mathrm{supp}\,\mathcal{S}) = \mathrm{supp}\,\mathcal{C}(\mathcal{S},\pi)$

$$\begin{aligned}\sigma_{\mathcal{R},\pi}(x) := \bigcap_{R \in \mathcal{R}\ \mathrm{s.t.}\ \ x \in \pi(R)} \pi(R) &\subset \bigcap_{R \in \mathcal{R}\ \mathrm{s.t.}\ \ x \in \pi(R)} \pi(S_R) \\ &\subset \bigcap_{S \in \mathcal{S}\mathrm{s.t.}\ \ x \in \pi(S)} \pi(S) =: \sigma_{\mathcal{S},\pi}(x) \in \mathcal{C}(\mathcal{S},\pi).\end{aligned}$$ ∎

Recall that the *fiber* P_x of P along π at x is the projection of $P \cap \pi^{-1}(\{x\})$ on the space $\mathrm{Ker}(\pi)$ (see Figure 2). An important property of a chamber complex is that all fibers are normally equivalent in each relative interior of cells of the chamber complex. More precisely, let $\sigma \in \mathcal{C}(P,\pi)$ be a chamber, and x and x' two points in its

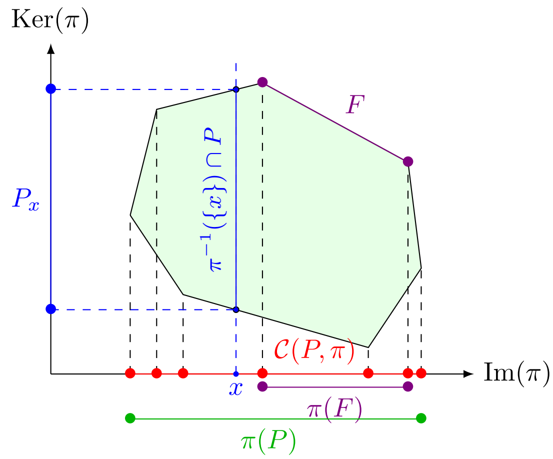


Figure 2: A polytope P and its projection $\pi(P)$ in green, its chamber complex in red on the x-axis and a fiber P_x in blue on the y-axis, for the orthogonal projection π on the horizontal axis, a face F and its projection $\pi(F)$ in purple.

relative interior, then, P_x and $P_{x'}$ are *normally equivalent*, see [3]. Thus, we define the normal fan $\mathcal{N}_\sigma$ above[4] $\sigma \in \mathcal{C}(P,\pi)$ by:

$$\mathcal{N}_\sigma := \mathcal{N}(P_x) \quad \text{for an arbitrary } x \in \text{ri}(\sigma).$$

The terms *parametrized polyhedron*, instead of fibers, and *validity domains*, instead of chambers, are also used in the literature [10, 35].

3. Exact quantization of the 2-stage problem. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $\boldsymbol{c} \in L^1(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^m)$ be an integrable random vector, and suppose $\xi = (A, B, b)$ is deterministic. We study the expected cost-to-go function of the 2-stage stochastic linear problem, written as

$$V(x) := \mathbb{E}\left[\hat{V}(x,\boldsymbol{c})\right] \quad \text{with} \quad \hat{V}(x,c) := \min_{y \in \mathbb{R}^m} \; c^\top y \quad \text{s.t.} \quad Ay + Bx \leqslant b. \tag{3.1}$$

The dual of the latter problem, for given x and c, is

$$\begin{aligned} \max_{\lambda \in \mathbb{R}^\ell} \quad & (Bx - b)^\top \lambda \\ \text{s.t.} \quad & A^\top \lambda = -c, \\ & \lambda \geqslant 0. \end{aligned} \tag{3.2}$$

We denote the *coupling constraint polyhedron* of Problem (3.1) by

$$P := \{(x,y) \in \mathbb{R}^{n+m} \mid Ay + Bx \leqslant b\},$$

and π the projection of $\mathbb{R}^n \times \mathbb{R}^m$ onto $\mathbb{R}^n$ such that $\pi(x,y) = x$.

[4]The normal fan $\mathcal{N}_\sigma \subset 2^{\text{Ker}(\pi)}$ above σ should not be confused with $\mathcal{N}(\sigma) \subset 2^{\text{Im}(\pi)}$ the normal fan of σ which will never appear in this paper.

The projection of P is the following polyhedron:

$$\pi(P) = \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m,\ Ay + Bx \leqslant b\}, \tag{3.3}$$

and for any $x \in \mathbb{R}^n$, the *fiber* of P along π is

$$P_x := \{y \in \mathbb{R}^m \mid Ay + Bx \leqslant b\}. \tag{3.4}$$

3.1. Chamber complexes arising from 2-stage problems. The following lemma provides an explicit formula for the cost-to-go function. It shows that an optimal recourse can be chosen as a function of c that is piecewise constant on the normal fan of P_x.

LEMMA 3.1. *Let $x \in \mathbb{R}^n$ and $c \in \mathbb{R}^m$,*
1. *If $x \notin \pi(P)$, then $\hat{V}(x,c) = +\infty$;*
2. *If $x \in \pi(P)$ and $-c \notin \mathrm{Cone}(A^\top)$, then $\hat{V}(x,c) = -\infty$;*
3. *Suppose now that $x \in \pi(P)$ and $-c \in \mathrm{Cone}(A^\top)$. For each cone $N \in \mathcal{N}(P_x)$, let us select in an arbitrary manner a vector c_N in $\mathrm{ri}(-N)$. Then, there exists a vector $y_N(x)$ which achieves the minimum in the expression of $\hat{V}(x,c_N)$ in (3.1), independently of the choice of $c_N \in ri(-N)$. Further, for any selection of such a $y_N(x)$, we have*

$$\hat{V}(x,c) = \sum_{N \in \mathcal{N}(P_x)} \mathbb{1}_{c \in -\operatorname{ri} N}\ c^\top y_N(x)\ . \tag{3.5}$$

Proof. The first point comes from the definitions of $\pi(P)$ in (3.3) and $\hat{V}(x,c)$ in (3.1). If $x \in \pi(P)$ and $-c \notin \mathrm{Cone}(A^\top)$, then the primal problem (3.1) is feasible and the dual problem is (3.2) infeasible. Thus, by strong duality, $\hat{V}(x,c) = -\infty$.

By (2.2), we have that $\big(\mathrm{rc}(P_x)\big)^\circ = \operatorname{supp} \mathcal{N}(P_x)$. Further, by (2.1) all non-empty fibers P_x have the same recession cone $\{y \mid Ay \leqslant 0\}$ whose polar is $\mathrm{Cone}(A^\top)$.

Assume now that $x \in \pi(P)$ and $-c \in \mathrm{Cone}(A^\top) = \mathrm{supp}(\mathcal{N}(P_x))$. Then, there exists $N \in \mathcal{N}(P_x)$ such that $-c \in \mathrm{ri}(N)$. Moreover, for every choice of $c_N \in -\mathrm{ri}(N)$, we have $\arg\min_{y \in P_x} c^\top y = \arg\min_{y \in P_x} c_N^\top y$, see *e.g.*, [36, Cor. 1(c)]. Moreover, there exists $y_N(x)$ such that $N = N_{P_x}\big(y_N(x)\big)$ by definition of a normal cone, thus $y_N(x) \in \arg\min_{y \in P_x} c_N^\top y$; in particular, the latter $\arg\min$ is non-empty. Thus, when $-c \in \mathrm{ri}(N)$, $\hat{V}(x,c) = c^\top y_N(x)$.

Thanks to the partition property of Proposition 2.2, we know that c belongs to the relative interior of precisely one cone in the normal fan of P_x, in particular $1 = \sum_{N \in \mathcal{N}(P_x)} \mathbb{1}_{c \in -\operatorname{ri} N}$ leading to (3.5). ∎

Having this property in mind, we make the following assumption:

ASSUMPTION 1. *The cost $\boldsymbol{c} \in L^1(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^m)$ is integrable with $\boldsymbol{c} \in -\mathrm{Cone}(A^\top)$ almost surely.*

THEOREM 3.2 (Local, uniform quantizations of the cost distribution). *Let $x \in \pi(P)$, and σ be a cell of $\mathcal{C}(P,\pi)$ the chamber complex of the coupling constraint polyhedron P along the projection π on the x-space. Assume that $x \in \mathrm{ri}(\sigma)$.*

Under Assumption 1, for every refinement $\mathcal{R}$ of $-\mathcal{N}_\sigma$, we have:

$$V(x) = \sum_{R \in \mathcal{R}} \check{p}_R \hat{V}(x, \check{c}_R) \quad \textit{with} \quad \hat{V}(x,\check{c}_R) := \min_{y \in \mathbb{R}^m} \quad \check{c}_R^\top y + \mathbb{I}_{Ay+Bx \leqslant b}. \tag{3.6}$$

where $\check{p}_R := \mathbb{P}\big[\boldsymbol{c} \in \mathrm{ri}(R)\big]$ and $\check{c}_R := \mathbb{E}\big[\boldsymbol{c} \,|\, \boldsymbol{c} \in \mathrm{ri}(R)\big]$ if $\check{p}_R > 0$ and $\check{c}_R := 0$ if $\check{p}_R = 0$.

In particular, if $\mathcal{R}$ is a refinement of $\bigwedge_{\sigma \in \mathcal{C}(P,\pi)} -\mathcal{N}_\sigma$, (3.6) holds for all $x \in \pi(P)$.

This is an exact quantization result, since (3.6) shows that $V(x)$ coincides with the value function of a second stage problem with a cost distribution supported by the finite set $\{\check{c}_R \mid R \in \mathcal{R}\}$.

Proof. Let $\sigma \in \mathcal{C}(P, \pi)$ and $x \in \mathrm{ri}(\sigma)$ then, by definition, $\mathcal{N}(P_x) = \mathcal{N}_\sigma$.

For $R \in \mathcal{R}$, there exists one and only one $N \in -\mathcal{N}_\sigma$ such that $\mathrm{ri}(R) \subset \mathrm{ri}(N)$, that we denote $N(R)$. Indeed, as $\mathcal{R}$ is a refinement of $-\mathcal{N}_\sigma$, there exists at least one, and as $-\mathcal{N}_\sigma$ is a polyhedral complex it is unique.

By Lemma 3.1, under Assumption 1 and since $x \in \pi(P)$,

$$
\begin{aligned}
V(x) &= \mathbb{E}\Big[\sum_{N\in\mathcal{N}(P_x)} \mathbb{1}_{\boldsymbol{c}\in -\mathrm{ri}\, N} \boldsymbol{c}^\top y_N(x)\Big] \\
&= \mathbb{E}\Big[\sum_{N\in-\mathcal{N}_\sigma} \sum_{R\in\mathcal{R} \mid \mathrm{ri}(R)\subset \mathrm{ri}(N)} \mathbb{1}_{\boldsymbol{c}\in \mathrm{ri}\, R}\, \boldsymbol{c}^\top y_N(x)\Big] \quad \text{by the partition property,} \\
&= \sum_{R\in\mathcal{R}} \mathbb{E}\big[\mathbb{1}_{\boldsymbol{c}\in\mathrm{ri}\, R}\boldsymbol{c}^\top\big] y_{N(R)}(x) \quad \text{by linearity,} \\
&= \sum_{R\in\mathcal{R}} \check{p}_R \check{c}_R^\top y_{N(R)}(x), \\
&= \sum_{R\in\mathcal{R}} \check{p}_R \min_{y\in\mathbb{R}^m} \check{c}_R^\top y + \mathbb{I}_{Ay+Bx\leqslant b},
\end{aligned}
$$

the last equality is by definition of $y_{N(R)}(x)$ as $\check{c}_R \in N(R)$, which leads to (3.6). ◻

Note that $\mathcal{R} = \bigwedge_{\sigma\in\mathcal{C}^{\max}(P,\pi)} -\mathcal{N}_\sigma$ satisfies the condition of Theorem 3.2 since if τ is a face of σ in the chamber complex, $\mathcal{N}_\sigma$ refines $\mathcal{N}_\tau$ by [44, Lemma 2.2].

3.2. Illustrative example and analytical formulas. In this section, we illustrate the exact quantization result on an example, for different distributions. To apply this result, we need to compute the quantized costs and probabilities $\check{c}_R$ and $\check{p}_R$ arising in Theorem 3.2. This can be done exactly for uniform, exponential and Gaussian distributions. The formulas of quantized probabilities and costs are summed up in Table 1. They rely on the exponential valuation of a simplicial cone (see [9] or [1, (8.2.2)]) in the exponential case, and on solid angles [45] for Gaussians (see [19] for details). We only provide these formulas for *simplices* or *simplicial cones* S with $\dim(S) = \dim(\mathrm{supp}\, \boldsymbol{c})$. This extends to any polyhedron R, through triangulation of $R \cap \mathrm{supp}(\boldsymbol{c})$ into simplices and simplicial cones $(S_k)_{k\in[l]}$. We then compute $\check{p}_R = \sum_{k=1}^l \check{p}_{S_k}$ and $\check{c}_R = \sum_{k=1}^l \check{p}_{S_k}\check{c}_{S_k}/\check{p}_R$ if $\check{p}_R \neq 0$ and $\check{c}_R = 0$ otherwise. Moreover, in [32], Lasserre showed analytical formulas to integrate polynomials on a simplex which open the door to formulas for distributions with polynomial densities, such as the Beta distribution. The approximation of the quantized costs and probabilities for general distributions is treated in subsection 5.2.

Consider the following second-stage problem, with $n = 1$ and $m = 2$:

$$
V(x) = \mathbb{E}\begin{bmatrix} \min\limits_{y\in\mathbb{R}^2} & \boldsymbol{c}^\top y \\ \text{s.t.} & \|y\|_1 \leqslant 1, \quad y_1 \leqslant x \text{ and } y_2 \leqslant x \end{bmatrix} .
$$

The coupling polyhedron is $P = \{(x, y) \in \mathbb{R}\times\mathbb{R}^2 \mid \|y\|_1 \leqslant 1,\, y_1 \leqslant x,\, y_2 \leqslant x\}$ presented in Figure 3, and its V-representation is the collection of vertices $(0, -1, 0)$,

	Uniform	Exponential	Gaussian
$d\mathbb{P}(c)$	$\frac{\mathbb{1}_{c\in Q}}{\mathrm{Vol}_d(Q)} d\mathcal{L}_{\mathrm{Aff}(Q)}(c)$	$\frac{e^{\theta^\top c}\mathbb{1}_{c\in K}}{\Phi_K(\theta)} d\mathcal{L}_{\mathrm{Aff}(K)}(c)$	$\frac{e^{-\frac{1}{2}c^\top M^{-2}c}}{(2\pi)^{\frac{m}{2}}\det M} dc$
$\operatorname{supp} \boldsymbol{c}$	Polytope: Q	Cone: K	$\mathbb{R}^m$
$\check{p}_S$	$\frac{\mathrm{Vol}_d(S)}{\mathrm{Vol}_d(Q)}$	$\frac{\lvert\det(\mathrm{Ray}(S))\rvert}{\Phi_K(\theta)} \prod_{r\in\mathrm{Ray}(S)} \frac{1}{-r^\top\theta}$	$\mathrm{Ang}\left(M^{-1}S\right)$
$\check{c}_S$	$\frac{1}{d+1}\sum_{v\in\mathrm{Vert}(S)} v$	$\left(\sum_{r\in\mathrm{Ray}(S)} \frac{-r_i}{r^\top\theta}\right)_{i\in[m]}$	$\frac{\sqrt{2}\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})} M \,\mathrm{SpCtr}\left(S\cap\mathbb{S}_{m-1}\right)$

Table 1: Probabilities $\check{p}_S$ and expectations $\check{c}_S$ arising from different cost distributions over simplicial cones or simplices $S \subset \operatorname{supp}(\boldsymbol{c})$ with $\dim S = \dim(\operatorname{supp}\boldsymbol{c})$, where $\mathcal{L}_A$ is the Lebesgue measure on an affine space A. We denote by $\mathrm{Vert}(S)$ the set of extreme points of a simplex S and by $\mathrm{Ray}(S)$ a collection of arbitrary representatives of extreme rays of a simplicial cone S. We denote by $\Phi_P(\theta) := \int_P e^{\theta^\top c} d\mathcal{L}_{\mathrm{Aff}(P)}(c)$ the *exponential valuation* of P with parameter θ, (see [1]). The solid angle is denoted by Ang and the spherical centroid by SpCtr (see [45]).

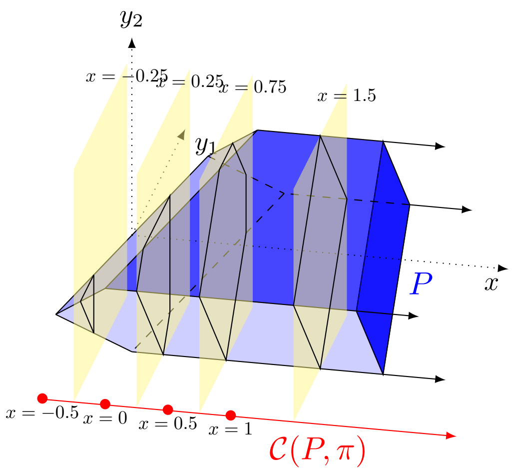
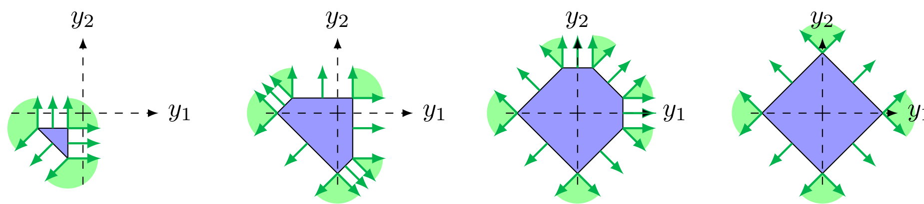


Figure 3: The coupling polyhedron P in blue, different cuts and fibers P_x vertical in yellow, and its chamber complex $\mathcal{C}(P, \pi)$ in red on the bottom.

$(-0.5, -0.5, -0.5)$, $(0, 0, -1)$, $(1, 1, 0)$, $(0.5, 0.5, 0.5)$, $(1, 0, 1)$ and the ray $(1, 0, 0)$. By projecting the different faces, we see that its projection is the half-line, $\pi(P) = [-0.5, +\infty)$ and its chamber complex $\mathcal{C}(P, \pi)$ is the collection of cells composed of $\{-0.5\}$, $[-0.5, 0]$, $\{0\}$, $[0, 0.5]$, $\{0.5\}$, $[0.5, 1]$, $\{1\}$, $[1, +\infty)$ as presented in Figure 3. As there are 4 different maximal chambers, there are 4 different classes of normally equivalent fibers as shown in Figure 4.

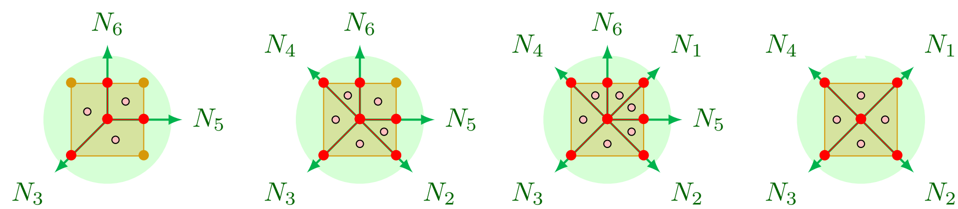
We evaluate $\check{c}_N$ and $\check{p}_N$ for $N \in -\mathcal{N}_\sigma$ using the formulas of Table 1. For example, when $\boldsymbol{c}$ is uniform on the centered ball for the ∞-norm of radius R, Figure 5 shows the regions of which the areas and centroids need to be computed. We sum up V in Figure 6 and present its value in Table 2 for different distributions.

3.3. Weighted fiber polyhedron. In this section, we provide an explicit representation of the expected cost-to-function in terms of the support function of a



(a) $x = -0.25$, $\sigma = [-0.5, 0]$ (b) $x = 0.25$, $\sigma = [0, 0.5]$ (c) $x = 0.75$, $\sigma = [0.5, 1]$ (d) $x \geqslant 1$, $\sigma = [1, +\infty)$

Figure 4: Fibers P_x in blue and their normal fan $\mathcal{N}(P_x) = \mathcal{N}_\sigma$ in green for various x.



(a) $\sigma = [-0.5, 0]$ (b) $\sigma = [0, 0.5]$ (c) $\sigma = [0.5, 1]$ (d) $\sigma = [1, +\infty)$

Figure 5: Exact quantization illustrated. The normal fan $\mathcal{N}_\sigma$ in green with $N_i = W_i^\top \mathbb{R}^+$, $\boldsymbol{c}$ is uniform on the support $Q = -Q = B_\infty(0, R)$ in light orange, the sets $W_i^\top \mathbb{R}^+ \cap Q$ in red. The polyhedral complex $\mathcal{R}_\sigma$ shown in red or orange. The quantized costs $\check{c}_N$ are determined by centroids (small circles in pink).

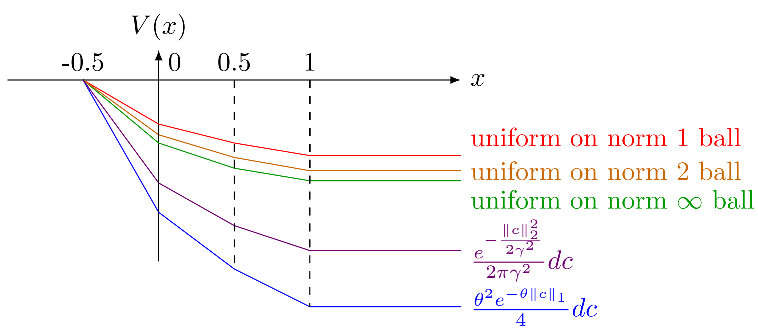


Figure 6: Graph of function V for various distribution of $\boldsymbol{c}$ with $R = \theta = \gamma = 1$.

weighted generalization of the notion of fiber polytope.

In [3], given a polytope P and its image $Q = \pi(P)$ under a linear projection mapping π, Billera and Sturmfels defined the *fiber polytope* of P over Q as the normalized Minkowski integral $\frac{1}{\mathrm{Vol}(Q)} \int_Q P_x dx$ of bounded fibers P_x (defined in (3.4)) where x is uniformly distributed on the polytope Q. We now extend the notion of fiber polytope. First, we allow the fibers to be polyhedron with non trivial recession cones and

$d\mathbb{P}(c)$	$-0.5 \leqslant x \leqslant 0$	$0 \leqslant x \leqslant 0.5$	$0.5 \leqslant x \leqslant 1$	$1 \leqslant x$
$\frac{\mathbb{1}_{\|c\|_1 \leqslant R}}{2R^2} dc$	$\frac{-7R}{24}(1+2x)$	$\frac{-R}{24}(7+6x)$	$\frac{-R}{6}(2+x)$	$\frac{-R}{2}$
$\frac{\theta^2 e^{-\theta\|c\|_1}}{4} dc$	$\frac{-7}{8\theta}(1+2x)$	$\frac{-1}{8\theta}(7+6x)$	$\frac{-1}{2\theta}(2+x)$	$\frac{-3}{2\theta}$
$\frac{\mathbb{1}_{\|c\|_\infty \leqslant R}}{4R^2} dc$	$\frac{-R}{12}(5+10x)$	$\frac{-R}{12}(5+4x)$	$\frac{-R}{6}(3+x)$	$\frac{-2R}{3}$
$\frac{e^{-\|c\|_2^2/2\gamma^2}}{2\pi\gamma^2} dc$	$\frac{-\gamma(2+\sqrt{2})(1+2x)}{2\sqrt{2\pi}}$	$\frac{-\gamma(2+\sqrt{2}+2\sqrt{2}x)}{2\sqrt{2\pi}}$	$\frac{-2\gamma(1+(-1+\sqrt{2})x)}{\sqrt{2\pi}}$	$-\frac{2}{\sqrt{\pi}}\gamma$
$\frac{\mathbb{1}_{\|c\|_2 \leqslant R}}{\pi R^2} dc$	$\frac{-R(2+\sqrt{2})(1+2x)}{3\pi}$	$\frac{-R(2+\sqrt{2}+2\sqrt{2}x)}{3\pi}$	$\frac{-4R(1+(-1+\sqrt{2})x)}{3\pi}$	$-\frac{4\sqrt{2}R}{3\pi}$

Table 2: Different values of $V(x)$ for different distributions of the cost $\boldsymbol{c}$.

lineality spaces. Secondly, we replace the uniform distribution on a polytope by a probability distribution on a polyhedron. We call this new polyhedron the *weighted fiber polyhedron*. To link this notion with stochastic programming, we give the definition with respect to the dual fibers D_c. We denote by $D_c := \{\lambda \in \mathbb{R}_+^\ell \mid A^\top \lambda + c = 0\}$ the admissible dual set for a fixed cost $c \in -\operatorname{Cone}(A)$, see (3.2).

Definition 3.3 (Weighted fiber polyhedron). *Let Assumption 1 holds. The* weighted fiber polyhedron E *of the bundle* $(D_c)_{c \in \operatorname{supp}(\boldsymbol{c})}$ *is the Minkowski integral of all the fibers at* c *when* c *varies according to its probability distribution:*

$$E := \mathbb{E}\big[D_{\boldsymbol{c}}\big] = \int D_c \mathbb{P}(dc) = \Big\{ \int \lambda(c)\mathbb{P}(dc) \;\Big|\; \lambda(c) \in D_c \, \mathbb{P}\text{-}a.s.,\ \lambda \in L^1(\mathbb{P}, \mathbb{R}^m, \mathbb{R}^\ell) \Big\}.$$

Note that, when $\mathbb{P}$ is a uniform probability measure on a polytope, we recover the original fiber polytope. The weighted fiber polyhedron is indeed a polyhedron as, by [3, Theorem 1.5], we can replace the Minkowski integral by a finite Minkowski, leveraging the normal equivalence of the fibers on the cells of the chamber complex. More precisely, let $D := \{(\lambda, c) \in \mathbb{R}^\ell \times \mathbb{R}^m \mid A^\top \lambda + c = 0, \lambda \geqslant 0\}$ be the dual coupling polyhedron, and $\pi_c^{\lambda,c}$ the orthogonal projection of $\mathbb{R}^\ell \times \mathbb{R}^m$ to $\mathbb{R}^m$. Recall that $\mathcal{C}(D, \pi_c^{\lambda,c})$ denotes the chamber complex of D along $\pi_c^{\lambda,c}$. We have

$$E = \sum_{\gamma \in \mathcal{C}(D, \pi_c^{\lambda,c})} \check{p}_\gamma D_{\check{c}_\gamma}. \tag{3.7}$$

where $\check{p}_\gamma := \mathbb{P}\big[\boldsymbol{c} \in \operatorname{ri}(\gamma)\big]$ and $\check{c}_\gamma := \mathbb{E}\big[\boldsymbol{c} \mid \boldsymbol{c} \in \operatorname{ri}(\gamma)\big]$ is the centroid of the cell γ if $\check{p}_\gamma > 0$ and $\check{c}_\gamma$ is an arbitrary point in $\operatorname{ri}(\gamma)$ if $\check{p}_\gamma = 0$.

The weighted fiber polyhedron synthesizes the polyhedral structure of 2SLP with stochastic cost $\boldsymbol{c}$. In particular, the expected cost-to-go function V is, up to an affine transformation, equal to the support function of the weighted fiber polyhedron.

Theorem 3.4. *Let Assumption 1 holds. Then, the expected cost-to-go* V *defined in* (3.1) *is the composition of the support function* σ_E *of the weighted fiber polyhedron* E *defined in Definition 3.3 and the affine transformation* $a : x \mapsto Bx - b$

$$V(x) = \sigma_E \circ a(x) := \sup_{\lambda \in E} (Bx - b)^\top \lambda.$$

In particular, the affine regions of V *are exactly the maximal cells of the polyhedral complex* $a^{-1}\Big(\mathcal{N}(E)\Big)$.

The proof consists in applying the interchangeability theorem (see [46, Thm 14.60]) to the dual formulation of the second stage problem.

Proof. Under Assumption 1, we have $\boldsymbol{c} \in -\operatorname{Cone}(A^\top)$ almost surely, thus for $x \in \mathbb{R}^n$,

$$
\begin{aligned}
V(x) &= \mathbb{E}\big[\hat{V}(x, \boldsymbol{c})\big], \\
&= \mathbb{E}\big[\sup_{\lambda \in \mathbb{R}^\ell} (Bx - b)^\top \lambda - \mathbb{I}_{\lambda \in D_{\boldsymbol{c}}}\big] && \text{by (3.2)}, \\
&= \int_{-\operatorname{Cone}(A^\top)} \sup_{\lambda \in \mathbb{R}^\ell} \Big((Bx - b)^\top \lambda - \mathbb{I}_{\lambda \in D_c}\Big) \mathbb{P}(dc), \\
&= \sup_{\lambda(.) \in L^1(\mathbb{P}, \mathbb{R}^n, \mathbb{R}^\ell)} \int_{-\operatorname{Cone}(A^\top)} \Big((Bx - b)^\top \lambda(q) - \mathbb{I}_{\lambda(c) \in D_c}\Big) \mathbb{P}(dc).
\end{aligned}
$$

Indeed, we can apply [46, Thm 14.60] since the opposite of the function $(c, \lambda) \mapsto (Bx - b)^\top \lambda - \mathbb{I}_{\lambda \in D_c}$ is a normal integrand (see [46, Def 14.27]) and $L^1(\mathbb{P}, \mathbb{R}^n, \mathbb{R}^\ell)$ is a decomposable space (see [46, Def 14.59]) with the measure $\mathbb{P}$. Thus,

$$
\begin{aligned}
V(x) &= \sup_{\lambda(.) \in L^1(\mathbb{P}, \mathbb{R}^n, \mathbb{R}^\ell)} (Bx - b)^\top \int_{-\operatorname{Cone}(A^\top)} \lambda(c) \mathbb{P}(dc) - \mathbb{I}_{\lambda(c) \in D_c \ \mathbb{P}\text{- a.s.}}, \\
&= \sup_{\lambda(.) \in L^1(\mathbb{P}, \mathbb{R}^n, \mathbb{R}^\ell) \,\big|\, \lambda(c) \in D_c \ \mathbb{P}\text{- a.s.}} (Bx - b)^\top \int_{-\operatorname{Cone}(A^\top)} \lambda(c) \mathbb{P}(dc), \\
&= \sup_{\lambda \in E} (Bx - b)^\top \lambda.
\end{aligned}
$$

∎

Remark 3.5 (Links between uniform exact quantization and secondary fan). *We can retrieve the uniform exact quantization Theorem 3.2, in a dual formulation, from Theorem 3.4 and from the decomposition as a Minkowski sum in (3.7). Note that the weighted fiber polyhedron is not* universal *as it determines exactly the affine regions of the expected cost-to-go function, for a given cost distribution, and not only a refinement. However, there exists an explicit and* universal *fan,* i.e., *independent of the distribution of* $\boldsymbol{c}$*, which refines* $\mathcal{N}(E)$*. More precisely, we have*

$$
-\Sigma\text{-fan}(A^\top) \preccurlyeq \mathcal{N}(E) \tag{3.9}
$$

where $\Sigma\text{-fan}(A^\top)$*, is the so-called* secondary fan*, defined in [13, 5.2.11]. It is the normal fan of a well-studied polytope called* secondary polytope *introduced in [22] (see also [13, Section 5]). Note that the secondary polytope is a special case of fiber polytope ([3]).*

Further, through technical, yet basic, computations, we also have that

$$
\mathcal{C}(P, \pi) = a^{-1}(-\Sigma\text{-fan}(A^\top)). \tag{3.10}
$$

In particular, while providing a more precise characterization of the affine regions, (3.9) and (3.10) together with Theorem 3.4 show that the cells of the chamber complex are universal *affine regions. A result we establish in Theorem 3.6 by a more elementary way.*

However, to extend these results to the multistage setting, we would need a more substantial generalization of fiber polytopes, taking into account nonanticipativity constraints and the nested structure of the control problem. We discuss such a generalization in [19]. In section 4, we develop a more direct approach to the multistage problem, in terms of chamber complexes.

3.4. Explicit characterization of expected cost-to-go. As a consequence of the exact quantization Theorem 3.2, we obtain explicit representations for the values and subdifferentials of the expected cost-to-go function V. We also show that V is affine on every cell of the chamber complex for every distribution of the random cost.

THEOREM 3.6 (Characterization of the expected cost-to-go function). *Let Assumption 1 holds. For $x \in \mathbb{R}^n$ and $c \in \mathbb{R}^m$, we denote*

$$D_c^{b-Bx} := \operatorname{argmax}\big\{(Bx-b)^\top \lambda : A^\top \lambda = -c, \lambda \geqslant 0\big\},$$

the set of optimal dual solutions of the second stage problem. Then,

$$\forall \sigma \in \mathcal{C}(P,\pi), \quad \forall x, x' \in \operatorname{ri}(\sigma), \quad \forall c \in \operatorname{supp}(\boldsymbol{c}), \qquad D_c^\sigma := D_c^{b-Bx} = D_c^{b-Bx'}.$$

Set

$$\alpha_\sigma := \sum_{N \in -\mathcal{N}_\sigma} B^\top \lambda_{\check{c}_N}^\sigma \quad \textit{and} \quad \beta_\sigma := \sum_{N \in -\mathcal{N}_\sigma} -b^\top \lambda_{\check{c}_N}^\sigma,$$

where λ_c^σ is an element of D_c^σ. Then, we have

$$\forall \sigma \in \mathcal{C}(P,\pi), \quad \forall x \in \sigma, \quad V(x) = \alpha_\sigma^\top x + \beta_\sigma, \tag{3.11a}$$

$$\forall x \in \mathbb{R}^n, \quad V(x) = \mathbb{I}_{x \in \pi(P)} + \max_{\sigma \in \mathcal{C}^{\max}(P,\pi)} \alpha_\sigma^\top x + \beta_\sigma. \tag{3.11b}$$

In particular, for all distributions of $\boldsymbol{c}$ satisfying Assumption 1, V is affine on each cell of $\mathcal{C}(P,\pi)$, i.e. the cells of the chamber complex are universal *affine regions.*

Moreover, we characterize the subdifferential of the cost-to-go function as

$$\partial V(x) = N_{\pi(P)}(x) + \operatorname{Conv}\big\{(\alpha_\sigma)_{\sigma \in \mathcal{C}^{\max}(P,\pi) \,|\, x \in \sigma}\big\}.$$

Proof. By the basis decomposition theorem, see [53], we have that $D_c^\psi = D_c^{\psi'}$ for all ψ and ψ' belonging to the same relative interior of a cone of the secondary fan $\Sigma\text{-fan}(W^\top)$. In particular, by (3.10), for every x, x' in the same relative interior of a chamber σ, we have $D_c^{b-Bx} = D_c^{b-Bx'}$.

For all $x \in \operatorname{ri}(\sigma) \subset \pi(P)$ and all $c \in \operatorname{supp}(\boldsymbol{c})$, by Lemma 3.1, we have $\hat{V}(x,c) < +\infty$ and then by strong duality, $\hat{V}(x,c) = (Bx-b)^\top \lambda_\sigma^c$. Then by the exact quantization result (3.6), for all $x \in \operatorname{ri}(\sigma)$,

$$V(x) = \sum_{N \in -\mathcal{N}_\sigma} \check{p}_N \hat{V}(x, \check{c}_N) = \sum_{N \in -\mathcal{N}_\sigma} \check{p}_N (Bx-b)^\top \lambda_\sigma^{\check{c}_N} = \alpha_\sigma^\top x + \beta_\sigma.$$

Further, as V is lower semicontinuous and convex, we deduce (3.11a).

To show (3.11b), suppose first that $\dim\big(\pi(P)\big) = m$. Then, for $\sigma \in \mathcal{C}^{\max}(P,\pi)$, $x \to \alpha_\sigma^\top x + \beta_\sigma$ is a supporting affine function of V which coincide with V on σ whose dimension is m. Since $\bigcup_{\sigma \in \mathcal{C}^{\max}(P,\pi)} \sigma = \operatorname{supp}(\mathcal{C}(P,\pi)) = \pi(P)$, V is piecewise affine on the polyhedron $\pi(P)$ and equals to $+\infty$ elsewhere. Together with convexity of V, this yields (3.11b). When $\pi(P)$ is not full dimensional, we get the same result by restraining the ambient space to the affine hull $\operatorname{Aff}\big(\pi(P)\big)$. Since $\mathcal{C}(P,\pi)$ does not depend on $\boldsymbol{c}$, for all distributions of $\boldsymbol{c}$ satisfying *Assumption* 1, V is affine on each cell of $\mathcal{C}(P,\pi)$. Finally, the subgradient formula follows from (3.11). ∎

Remark 3.7. *Let $\mathcal{V}^{\max}$ be the collection of affine regions of V. Theorem 3.6 implies that the chamber complex $\mathcal{C}^{\max}(P,\pi)$ refines $\mathcal{V}^{\max}$. However, it does not imply that $\mathcal{C}^{\max}(P,\pi)=\mathcal{V}^{\max}$. Indeed, if $\boldsymbol{c}=0$ $\mathbb{P}$-almost surely, then $\mathcal{V}^{\max}=\{\pi(P)\}$.*

More precisely, for all cost distribution such that Assumption 1 holds, $\mathcal{V}^{\max}$ is the collection of maximal elements of a polyhedral complex $\mathcal{V}$ such that $\mathcal{C}(P,\pi)\preccurlyeq\mathcal{V}$. We gave an exact representation of $\mathcal{V}$ in Theorem 3.4, showing that $\mathcal{V}=a^{-1}\big(\mathcal{N}(E)\big)$.

4. Exact quantization of the multistage problem.

In this section, we show that the exact quantization result established above for a general cost distribution and deterministic constraints carries over to the case of stochastic constraints with finite support and then to multistage programming.

We denote by $\pi_x^{x,y}$ for the projection from $\mathbb{R}^n\times\mathbb{R}^m$ to $\mathbb{R}^n$ defined by $\pi_x^{x,y}(x',y')=x'$. The projections $\pi_{x,y}^{x,y,z}$, $\pi_x^{x,y,z}$, $\pi_y^{y,z}$, $\pi_{x_{t-1}}^{x_{t-1},z}$ are defined accordingly. Note that in the notation $\pi_x^{x,y,z}$, x, y and z are part of the notation and not parameters.

4.1. Propagating chamber complexes through Dynamic Programming.

We next show that chamber complexes are propagated through dynamic programming in a way that is universal with respect to the cost distribution. The following Lemma shows how to obtain (a refinement of) the affine regions of the cost-to-go function V_t. This refinement depends on the affine regions of V_{t+1} and not of the value of V_{t+1}[5].

Recall that, for a polyhedron P and a vector ψ, we denote $P^\psi:=\arg\min_{x\in P}\psi^\top x$. Let f be a polyhedral function on $\mathbb{R}^d$, with a slight abuse of notation we denote $\mathrm{epi}(f)^{\psi,1}=\arg\min_{(x,z)\in\mathrm{epi}(f)}\psi^\top x+z$. We denote $\mathcal{F}_{\mathrm{low}}\big(\mathrm{epi}(f)\big):=\{\mathrm{epi}(f)^{\psi,1}\mid\psi\in\mathbb{R}^d\}$ the set of *lower faces* of $\mathrm{epi}(f)$. The collection of projections (on $\mathbb{R}^d$) of lower faces of $\mathrm{epi}(f)$ is the coarsest polyhedral complex such that f is affine on each of its cells (see [13, Chapter 2]). Moreover, we have

$$\pi_{\mathbb{R}^d}\big(\mathrm{epi}(f)^{\psi,1}\big)=\arg\min_{x\in\mathbb{R}^d}\psi^\top x+f(x). \tag{4.1}$$

Lemma 4.1. *Let U be a polyhedral function on $\mathbb{R}^m$ and $\mathcal{U}:=\pi_y^{y,z}\Big(\mathcal{F}_{\mathrm{low}}\big(\mathrm{epi}(U)\big)\Big)$ a coarsest polyhedral complex such that U is affine on each element of $\mathcal{U}$. Let $\xi=(A,B,b)$ be fixed and Assumption 1 holds. Define, for all $x\in\mathbb{R}^n$*

$$Q(x,y):=U(y)+\mathbb{I}_{Ay+Bx\leqslant b},$$
$$V(x):=\mathbb{E}\big[\min_{y\in\mathbb{R}^m}\boldsymbol{c}^\top y+Q(x,y)\big].$$

Let $\mathcal{V}:=\mathcal{C}(\mathcal{F}(P)\wedge(\mathbb{R}^n\times\mathcal{U}),\pi_x^{x,y})\subset 2^{\mathbb{R}^n}$ with $P:=\{(x,y)\mid Ay+Bx\leqslant b\}$.

Then, $\mathcal{V}\preccurlyeq\mathcal{C}(\mathrm{epi}(Q),\pi_x^{x,y,z})$ and V is a polyhedral function which is affine on each element of $\mathcal{V}$.

Remark 4.2. *Thanks to a lift variable, we can rewrite the expected cost-to-go function as $V(x)=\mathbb{E}\big[\min_{y\in\mathbb{R}^m,z\in\mathbb{R}\,|\,(x,y,z)\in\mathrm{epi}(Q)}\boldsymbol{c}^\top y+z\big]$. A naive approach would be to apply directly Theorem 3.2 to this formulation as a 2SLP. However, in the multistage setting, $\mathrm{epi}(Q)$ depends on the latter random costs $\boldsymbol{c}_{t+1},\dots,\boldsymbol{c}_T$ and appears in the contraints. Thus, we cannot hope to obtain a universal polyhedral complex directly. We need the more subtle approach of Lemma 4.1 to show that the affine regions of V only depends on the affine regions of R, and on the coupling constraint polyhedron P and not on $\mathrm{epi}(Q)$.*

[5] In other words, the refinement obtained only depends on the projection of the lower faces of $\mathrm{epi}(V_{t+1})$ and not the whole epigraph.

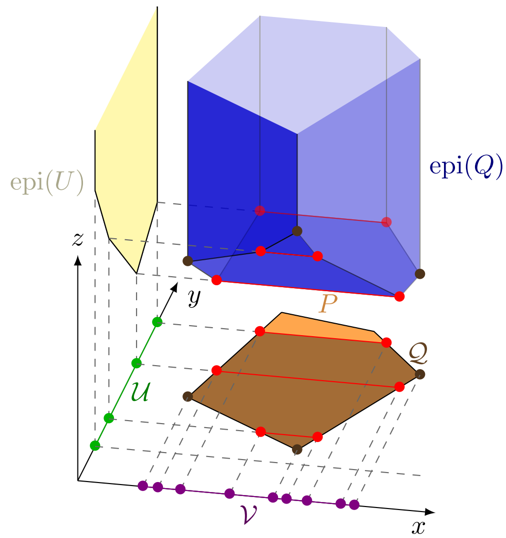


Figure 7: An illustration of the proof of Lemma 4.1: the epigraph $\mathrm{epi}(Q)$ of the coupling function in blue in the (x, y, z) space, the epigraph of U in yellow in the (y, z) plane, the affine regions $\mathcal{U}$ of U in green on the y axis, the coupling polyhedron P in orange and brown in the (x, y) plane, the polyhedral complex $\mathcal{Q}$ in red and brown in the (x, y) plane and the chamber complex $\mathcal{V}$ in violet on the x axis.

Proof. We have $\mathrm{epi}(Q) = \big(\mathbb{R}^n \times \mathrm{epi}(U)\big) \cap (P \times \mathbb{R}) \subset \mathbb{R}^{n+m+1}$ (see Figure 7). Since

$$V(x) = \mathbb{E}\big[\min_{y\in\mathbb{R}^m, z\in\mathbb{R}} \boldsymbol{c}^\top y + z + \mathbb{I}_{(x,y,z)\in \mathrm{epi}(Q)}\big],$$

by Theorem 3.6 applied to the problem with variables (y, z) and the coupling polyhedron $\mathrm{epi}(Q)$, V is a polyhedral function affine on each element of $\mathcal{C}(\mathrm{epi}(Q), \pi_x^{x,y,z})$. We now show that $\mathcal{V} \preccurlyeq \mathcal{C}(\mathrm{epi}(Q), \pi_x^{x,y,z})$. As $\mathrm{epi}(Q)$ is the epigraph of a polyhedral function, $\mathcal{Q} := \pi_{x,y}^{x,y,z}\big(\mathcal{F}_{\mathrm{low}}(\mathrm{epi}(Q))\big) \subset 2^{\mathbb{R}^{n+m}}$ is a polyhedral complex.

Let $\check{x} \in \pi_x^{x,y,z}(\mathrm{epi}(Q))$, using notation of Definition 2.6,

$$\begin{aligned}
\sigma_{\mathrm{epi}(Q),\pi_x^{x,y,z}}(\check{x}) &:= \bigcap_{F\in\mathcal{F}(\mathrm{epi}(Q))\,\text{s.t.}\ \ \check{x}\in\pi_x^{x,y,z}(F)} \pi_x^{x,y,z}(F),\\
&= \bigcap_{F\in\mathcal{F}_{\mathrm{low}}(\mathrm{epi}(Q))\,\text{s.t.}\ \ \check{x}\in\pi_x^{x,y,z}(F)} \pi_x^{x,y,z}(F),\\
&= \bigcap_{F'\in\mathcal{Q}\,\text{s.t.}\ \ \check{x}\in\pi_x^{x,y}(F')} \pi_x^{x,y}(F') =: \sigma_{\mathcal{Q},\pi_x^{x,y}}(\check{x}).
\end{aligned}$$

Indeed, as $\mathrm{epi}(Q)$ is an epigraph of a polyhedral function, if $F \in \mathcal{F}(\mathrm{epi}(Q))$ such that $\check{x} \in \pi_x^{x,y,z}(F)$ then there exists $G \in \mathcal{F}_{\mathrm{low}}(\mathrm{epi}(Q))$ such that $G \subset F$ and $\check{x} \in \pi_x^{x,y,z}(G)$, allowing us to go from the first to second equality. The third equality is obtained by setting $F' = \pi_{x,y}^{x,y,z}(F)$. Thus, $\mathcal{C}(\mathrm{epi}(Q), \pi_x^{x,y,z}) = \mathcal{C}(\mathcal{Q}, \pi_x^{x,y})$.

We now show that $\mathcal{F}(P) \wedge (\mathbb{R}^n \times \mathcal{U}) \preccurlyeq \mathcal{Q}$. Let $G \in \mathcal{F}(P) \wedge \big(\mathbb{R}^n \times \mathcal{U}\big)$. There exist $\sigma \in \mathcal{U}$ and $F \in \mathcal{F}(P)$ such that $G = F \cap (\mathbb{R}^n \times \sigma)$. By definition of $\mathcal{F}_{\mathrm{low}}$, there exists $\psi \in \mathbb{R}^m$ such that $\sigma = \pi_y^{y,z}\big(\,\mathrm{epi}(U)^{\psi,1}\big)$. We show that $G \subset \pi_{x,y}^{x,y,z}(\mathrm{epi}(Q)^{0,\psi,1}) \in \mathcal{Q}$.

Indeed, let $(x,y) \in G = F \cap \big(\mathbb{R}^n \times \pi_y^{y,z}(\operatorname{epi}(U)^{\psi,1})\big)$. We have $(x,y) \in F \subset P$ such that $y \in \arg\min_{y' \in \mathbb{R}^m} \big\{\psi^\top y' + U(y')\big\}$. Which implies that $(x,y) \in \arg\min \big\{\psi^\top y' + U(y') \mid (x',y') \in P\big\}$. This also reads, by (4.1), as $(x,y) \in \pi_{x,y}^{x,y,z}(\operatorname{epi}(Q)^{0,\psi,1})$. Thus, $G \subset \pi_{x,y}^{x,y,z}(\operatorname{epi}(Q)^{0,\psi,1}) \in \mathcal{Q}$ leading to $\mathcal{F}(P) \wedge (\mathbb{R}^n \times \mathcal{U}) \preccurlyeq \mathcal{Q}$. Finally, by monotonicity, Lemma 2.7 ends the proof. ∎

Remark 4.3. *In Lemma 4.1, the complex $\mathcal{V}$ is independent of the distribution of $\boldsymbol{c}$. However, for special choices of $\boldsymbol{c}$, V might be affine on each cell of a coarser complex than $\mathcal{V}$. For instance, if $U = 0$ and $\boldsymbol{c} \equiv 0$, we have that $V = \mathbb{I}_{\pi_x^{x,y}(P)}$, V is affine on $\pi_x^{x,y}(P)$. Nevertheless, $\mathcal{V} = \mathcal{C}(P, \pi_x^{x,y})$ is generally finer than $\mathcal{F}\big(\pi_x^{x,y}(P)\big)$. Note that the chambers of $\mathcal{V}$ can be enumerated thanks to the algorithm described in [10] (where chambers are called validity domains) or more generally by constructing the secondary polytope (see [2]).*

4.2. Exact quantization of MSLP. We next show that the multistage program with arbitrary cost distribution is equivalent to a multistage program with independent, finitely distributed, cost distributions. Further, for all step t, there exist affine regions, independent of the distributions of costs, where V_t is affine. Assumption 1 is naturally extended to the multistage setting as follows

Assumption 2. *The sequence $(\boldsymbol{c}_t, \boldsymbol{\xi}_t)_{2 \leqslant t \leqslant T}$ is independent.*[6] *Further, for each $t \in \{2, \dots, T\}$, $\boldsymbol{\xi}_t = (\boldsymbol{A}_t, \boldsymbol{B}_t, \boldsymbol{b}_t)$ is finitely supported, and $\boldsymbol{c}_t \in L^1(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^{n_t})$ is integrable with $\boldsymbol{c}_t \in -\operatorname{Cone}(\boldsymbol{A}_t^\top)$ almost surely.*

Note that Assumption 2 does not require independence between $\boldsymbol{c}_t$ and $\boldsymbol{\xi}_t$. Let $t \in [T]$. For any $\xi := (A, B, b) \in \operatorname{supp}(\boldsymbol{\xi}_t)$ we define the coupling polyhedron

$$P_t(\xi) := \{(x_{t-1}, x_t) \in \mathbb{R}^{n_{t-1}} \times \mathbb{R}^{n_t} \mid Ax_t + Bx_{t-1} \leqslant b\},$$

and consider, for $x_{t-1} \in \mathbb{R}^{n_{t-1}}$,

$$\widetilde{V}_t(x_{t-1}|\xi) := \mathbb{E}\Big[\min_{x_t \in \mathbb{R}^{n_t}} \boldsymbol{c}_t^\top x_t + V_{t+1}(x_t) + \mathbb{I}_{Ax_t + Bx_{t-1} \leqslant b} \mid \boldsymbol{\xi}_t = \xi\Big]. \tag{4.3}$$

Then, the cost-to-go function V_t is obtained by

$$V_t(x_{t-1}) = \sum_{\xi \in \operatorname{supp}(\boldsymbol{\xi}_t)} \mathbb{P}\big[\boldsymbol{\xi}_t = \xi\big] \widetilde{V}_t(x_{t-1} \mid \xi). \tag{4.4}$$

The next two theorems extend the quantization results of Theorem 3.2 to the multistage settings.

Theorem 4.4 (Affine regions independent of the cost). *Assume that $(\boldsymbol{\xi}_t)_{t \in [T]}$ is a sequence of independent, finitely supported, random variables. We define by induction $\mathcal{P}_{T+1} := \{\mathbb{R}^{n_T}\}$ and for $t \in \{2, \dots, T\}$*

$$\mathcal{P}_{t,\xi} := \mathcal{C}\big(\mathbb{R}^{n_t} \times \mathcal{P}_{t+1} \wedge \mathcal{F}\big(P_t(\xi)\big), \pi_{x_{t-1}}^{x_{t-1}, x_t}\big), \tag{4.5a}$$

$$\mathcal{P}_t := \bigwedge_{\xi \in \operatorname{supp} \boldsymbol{\xi}_t} \mathcal{P}_{t,\xi}.$$

Then, for all costs distributions $(\boldsymbol{c}_t)_{2 \leqslant t \leqslant T}$ such that $(\boldsymbol{c}_t, \boldsymbol{\xi}_t)_{2 \leqslant t \leqslant T}$ satisfies Assumption 2 and all $t \in \{2, \dots, T\}$, we have $\operatorname{supp}(\mathcal{P}_t) = \operatorname{dom}(V_t)$, and V_t is polyhedral and affine on each cell of $\mathcal{P}_t$.

[6]The results can be adapted to non-independent $\boldsymbol{\xi}_t$ as long as $\boldsymbol{c}_t$ is independent of $(\boldsymbol{c}_\tau)_{\tau < t}$ conditionally on $(\boldsymbol{\xi}_{\tau \leqslant t})$.

REMARK 4.5. *The definition of $\mathcal{P}_{t,\xi}$ as the induction equation (4.5a) is the same as the definition of $\mathcal{V}$ in Lemma 4.1 and illustrated in Figure 7, by taking $\mathcal{U} = \mathcal{P}_{t+1}$, $P = P_t(\xi)$, $x = x_{t-1}$ and $y = x_t$ (see also Figure 9 for a particular 3SLP example).*

Proof. We set for all $t \in \{2, \dots, T+1\}$, $\mathcal{V}_t := \pi_{x_{t-1}}^{x_{t-1},z}\big(\mathcal{F}_{\text{low}}\big(\operatorname{epi}(V_t)\big)\big)$ the affine regions of V_t. As $V_{T+1} \equiv 0$ is polyhedral and affine on $\mathbb{R}^{n_T}$, we have $\mathcal{P}_{T+1} = \mathcal{V}_{T+1}$. Assume now that for $t \in \{2, \dots, T\}$, V_{t+1} is polyhedral and $\mathcal{P}_{t+1}$ refines $\mathcal{V}_{t+1}$ (*i.e.*, V_{t+1} is affine on each cell $\sigma \in \mathcal{P}_{t+1}$).

By Lemma 4.1, $\widetilde{V}_t(\cdot|\xi)$, defined in (4.3), is affine on each cell of $\mathcal{C}(\mathbb{R}^{n_t} \times \mathcal{V}_{t+1} \wedge \mathcal{F}\big(P_t(\xi)\big), \pi_{x_{t-1}}^{x_{t-1},x_t})$ which is refined by $\mathcal{P}_{t,\xi} = \mathcal{C}(\mathbb{R}^{n_t} \times \mathcal{P}_{t+1} \wedge \mathcal{F}\big(P_t(\xi)\big), \pi_{x_{t-1}}^{x_{t-1},x_t})$ by induction hypothesis and Lemma 2.7. Thus, by (4.4), V_t is affine on each cell of $\mathcal{P}_t$. In particular, V_t is polyhedral and $\mathcal{P}_t := \bigwedge_{\xi \in \operatorname{supp} \boldsymbol{\xi}_t} \mathcal{P}_{t,\xi}$ refines $\mathcal{V}_t$. Backward induction ends the proof. ∎

By Lemma 4.1, we have that $\mathcal{P}_{t,\xi} \preccurlyeq \mathcal{C}(\operatorname{epi}\big(Q_t^\xi\big), \pi_{x_{t-1}}^{x_{t-1},x_t,z})$ where $Q_t^\xi(x_{t-1}, x_t) := V_{t+1}(x_t) + \mathbb{I}_{Ax_t + Bx_{t-1} \leqslant b_t}$. In particular, consider $\sigma \in \mathcal{P}_{t,\xi}$, then for all $x_{t-1} \in \operatorname{ri}(\sigma)$, all fibers $\operatorname{epi}(Q_t^\xi)_{x_{t-1}}$ are normally equivalent. We can then define $\mathcal{N}_{t,\xi,\sigma} := \mathcal{N}(\operatorname{epi}(Q_t^\xi)_{x_{t-1}})$ for an arbitrary $x_{t-1} \in \operatorname{ri}(\sigma)$.

The next result shows that we can replace the MSLP problem (1.2) by an equivalent problem with a discrete cost distribution.

THEOREM 4.6 (Exact quantization of the cost distribution, Multistage case). *Assume that $(\boldsymbol{\xi}_t)_{t \in [T]}$ is a sequence of independent, finitely supported, random variables. Then, for all costs distributions such that $(\boldsymbol{c}_t, \boldsymbol{\xi}_t)_{2 \leqslant t \leqslant T}$ satisfies Assumption 2, for all $t \in [T]$, all $x_{t-1} \in \mathbb{R}^{n_{t-1}}$ and all $\xi \in \operatorname{supp}(\boldsymbol{\xi}_t)$, we have a quantized version of (4.3):*

$$\widetilde{V}_t(x_{t-1}|\xi) = \sum_{N \in \mathcal{N}_{t,\xi}} \check{p}_{t,N|\xi} \min_{x_t \in \mathbb{R}^{n_t}} \Big\{ \check{c}_{t,N|\xi}^\top x_t + V_{t+1}(x_t) + \mathbb{I}_{Ax_t + Bx_{t-1} \leqslant b} \Big\}.$$

where $\mathcal{N}_{t,\xi} := \bigwedge_{\sigma \in \mathcal{P}_{t,\xi}} -\mathcal{N}_{t,\xi,\sigma}$ and for all $\xi \in \operatorname{supp}(\boldsymbol{\xi}_t)$ and $N \in \mathcal{N}_{t,\xi}$ we denote

$$\check{p}_{t,N|\xi} := \mathbb{P}\big[\boldsymbol{c}_t \in \operatorname{ri} N \mid \ \boldsymbol{\xi}_t = \xi\big],$$
$$\check{c}_{t,N|\xi} := \begin{cases} \mathbb{E}\big[\boldsymbol{c}_t \mid \boldsymbol{c}_t \in \operatorname{ri} N, \boldsymbol{\xi}_t = \xi\big] & \textit{if } \mathbb{P}\big[\boldsymbol{\xi}_t = \xi, \boldsymbol{x} \in \operatorname{ri} N\big] \neq 0 \\ 0 & \textit{otherwise} \end{cases}.$$

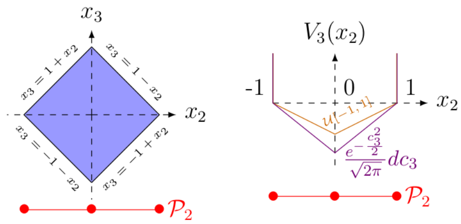
Proof. Since $\widetilde{V}_t(x_{t-1}|\xi) = \mathbb{E}\big[\min_{x_t \in \mathbb{R}^{n_t}, z \in \mathbb{R}} \boldsymbol{c}^\top x_t + z + \mathbb{I}_{(x_{t-1},x_t,z) \in \operatorname{epi}(Q_t^\xi)}\big]$ and $\mathcal{P}_{t,\xi}$ refines $\mathcal{C}(\operatorname{epi}\big(Q_t^\xi\big), \pi_{x_{t-1}}^{x_{t-1},x_t,z})$, by applying Theorem 3.2 with variables (x_t, z) and the coupling constraints polyhedron $\operatorname{epi}(Q_t^\xi)$, we deduce that the coefficients $(\check{p}_{t,N|\xi})_{N \in \mathcal{N}_{t,\xi}}$ and $(\check{c}_{t,N|\xi})_{N \in \mathcal{N}_{t,\xi}}$ satisfy

$$\widetilde{V}_t(x_{t-1}|\xi) = \sum_{N \in \mathcal{N}_{t,\xi}} \check{p}_{t,N|\xi} \min_{x_t \in \mathbb{R}^{n_t}, z \in \mathbb{R}} \Big\{ \check{c}_{t,N|\xi}^\top x_t + z + \mathbb{I}_{(x_{t-1},x_t,z) \in \operatorname{epi}(Q_t^\xi)} \Big\}.$$

as the deterministic coefficient before z is equal to its conditional expectation. ∎

In particular, the MSLP problem is equivalent to a finitely supported MSLP as shown in the following result.

For $t_0 \in [T-1]$, we construct the scenario tree $\mathcal{T}_{t_0}$ as follows. A node of depth $t - t_0$ of $\mathcal{T}_{t_0}$ is labeled by a sequence $(N_\tau, \xi_\tau)_{t_0 < \tau \leqslant t}$ where $N_\tau \in \mathcal{N}_{\tau,\xi_\tau}$ and $\xi_\tau \in \operatorname{supp}(\boldsymbol{\xi}_\tau)$. In this way, a node of depth $t - t_0$ of $\mathcal{T}_{t_0}$ keeps track of the sequence of realizations of the random variables $\boldsymbol{\xi}_\tau$ for times τ between t_0 and t, and of a selection of cones in



$d\mathbb{P}(c_3)$	$-1 \leqslant x_2 \leqslant 0$	$0 \leqslant x_2 \leqslant 1$
$\frac{\mathbb{1}_{-a\leqslant c_3\leqslant b}}{a+b}dc_3$	$\frac{a^2+b^2}{2(a+b)}(-1-x_2)$	$\frac{a^2+b^2}{2(a+b)}(-1+x_2)$
$\frac{e^{-c_3^2/2\sigma^2}}{\sigma\sqrt{2\pi}}dc_3$	$\frac{\sigma\sqrt{2}}{\sqrt{\pi}}(-1-x_2)$	$\frac{\sigma\sqrt{2}}{\sqrt{\pi}}(-1+x_2)$

Figure 8: The coupling constraint polyhedron P_3 and V_3 for two distributions of $\boldsymbol{c}_3$.

$\mathcal{N}_{t,\xi_t}$ at the same times. Note that, by the independence assumption, all the subtrees of $\mathcal{T}_{t_0}$, starting from a node of depth $t-t_0$ are the same as $\mathcal{T}_{t_0+t}$. We denote by $\mathrm{lv}(\mathcal{T}_{t_0})$ the set of leaves of $\mathcal{T}_{t_0}$.

Corollary 4.7 (Equivalent finite tree problem). *Define the quantized probability cost* $c_\nu := \check{c}_{t,N_t|\xi_t}$ *and probability* $p_\nu := \prod_{t_0<\tau\leqslant t} p_{\xi_\tau}\check{p}_{\tau,N_\tau|\xi_\tau}$, *for all nodes* $\nu = (N_\tau, \xi_\tau)_{t_0<\tau\leqslant t}$. *Then, the cost-to-go functions associated with (1.1) are given by*

$$V_{t_0}(x_0) = \min_{(x_\nu)_{\nu\in\mathcal{T}_{t_0}}} \sum_{\nu\in\mathcal{T}_{t_0}} p_\nu c_\nu^\top x_\nu$$

$$s.t. \quad Ax_\mu + Bx_\nu \leqslant b \qquad \forall \nu \in \mathcal{T}_{t_0} \setminus \mathrm{lv}(\mathcal{T}_{t_0}), \forall \mu \succ\!\!\!\!\!\;\neq \nu \ ,$$

for all $2 \leqslant t_0 \leqslant T-1$. *Here,* x_0 *is the value of* x *at the root node of* $\mathcal{T}_{t_0}$, *and the notation* $\forall \mu = (\nu, N, A, B, b) \succ\!\!\!\!\!\;\neq \nu$ *indicates that* μ *ranges over the set of children of* ν.

4.3. Illustrative example in 3SLP. We now illustrate the exact quantization result by considering the following three-stage stochastic linear problem:

$$\min_{x_1\in\mathbb{R}\,|\,x_1\in P_1} c_1 x_1 + \underbrace{\mathbb{E}\left[\min_{x_2\in\mathbb{R}\,|\,(x_1,x_2)\in P_2} \boldsymbol{c_2} x_2 + \underbrace{\mathbb{E}\left[\min_{x_3\in\mathbb{R}\,|\,(x_2,x_3)\in P_3} \boldsymbol{c_3} x_3\right]}_{V_3(x_2)}\right]}_{V_2(x_1)} \ .$$

with $P_2 = \{(x_1, x_2) \in \mathbb{R}^2 \,\big|\, -0.5 \leqslant x_2 \leqslant 1.3, 1 \leqslant x_1 - x_2 \leqslant 3\}$ and $P_3 = \{(x_2, x_3) \in \mathbb{R}^2 \,\big|\, \|(x_2, x_3)\|_1 \leqslant 2\}$. We compute V_3 (see Figure 8) and the chamber complex $\mathcal{P}_2$ composed of the cells $\{-1\}$, $[-1, 0]$, $\{0\}$, $[0, 1]$ and $\{1\}$.

Thanks to $\mathcal{P}_2$ and the coupling polyhedron P_2, we compute the chamber complex $\mathcal{P}_1$ whose chambers are $\{0.5\}$,$[0.5, 1]$,$\{1\}$,$[1, 2]$,$\{2\}$,$[2, 2.5]$,$\{2.5\}$,$[2.5, 3]$,$\{3\}$,$[3, 4]$ and $\{4\}$ (see Figure 9). We deduce the differents normal fans, for each chambers of $\mathcal{P}_1$ (see Figures 10 and 11).

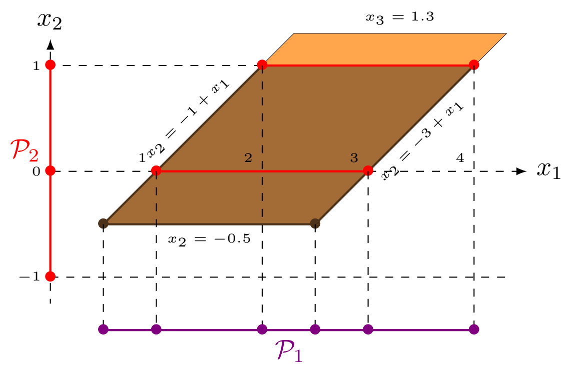


Figure 9: The coupling constraint polyhedron P_2, the chamber complexes $\mathcal{P}_2$ and $\mathcal{P}_1$

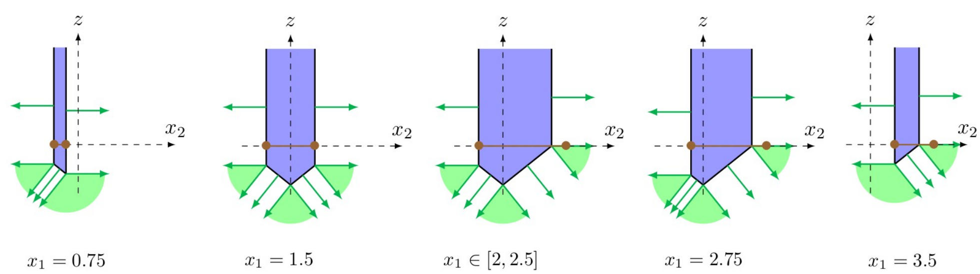
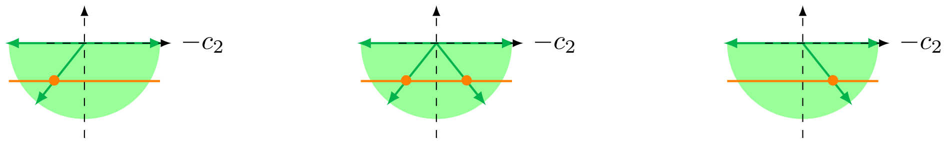


Figure 10: The fiber $E_{2,x_1} = \mathrm{epi}(V_3) \cap (P_{2,x_1} \times \mathbb{R})$ in blue, of the epigraph $E_2 := \mathrm{epi}(Q_2)$ where Q_2 is the polyhedral function $Q_2 : (x_1, x_2) \mapsto V_3(x_2) + \mathbb{I}_{(x_1,x_2) \in P_2}$ and P_{2,x_1} is in brown, its normal fan $\mathcal{N}(E_{2,x_1})$ in green for $\boldsymbol{c}_3$ following the standard normal distribution and different values of x_1.

5. Complexity. Hanasusanto, Kuhn and Wiesemann showed in [26] that 2-stage stochastic programming is ♯P-hard, by reducing the computation of the volume of a polytope to the resolution of a 2-stage stochastic program. Nevertheless, we show that for a fixed dimension of the recourse space, 2-stage programming is polynomial. Therefore, the status of 2-stage programming seems somehow comparable to the one of the computation of the volume of a polytope – which is also both ♯P-hard and polynomial when the dimension is fixed (see [33] or [23, 3.1.1]). Another example of ♯P-hard problems that are fixed dimension polynomial is the problem of counting the integer points in a given polytope (see [34]) We shall see that a similar result holds for multistage stochastic linear programming.

We first give a summary of our method. A naive approach would be to use directly the exact quantization result Theorem 3.2, for every x. However, even in the 2-stage case, the latter yields a linear program of an exponential size when only the recourse dimension m is fixed. Indeed, the size of the quantized linear program, $(2SLP)$ is polynomial only when *both* n and m are fixed. This is because $\bigwedge_{\sigma \in \mathcal{C}(P,\pi)} -\mathcal{N}_\sigma$ can have, by McMullen's and Stanley's upper bound theorems ([39, 52]), an exponential size in n and m, and these bounds are tight. Hence, to handle the case in which only the recourse dimension m is fixed, we need additional ideas. We use the quantization



(a) $x_1 \in [0.5, 1]$ (b) $x_1 \in [1, 3]$ (c) $x_1 \in [3, 4]$

Figure 11: The normal fan $\mathcal{N}(E_{2,x_1})$ in green, and its intersection with $\{-1\} \times \mathbb{R}$ in orange, for $\boldsymbol{c}_3$ following the standard normal distribution and different values of x_1.

result, Theorem 3.2 only for a *fixed* x, observing that when m is fixed, $\mathcal{N}(P_x)$ has a polynomial size. We thus have a polynomial time oracle that gives the values $V(x)$ by Theorem 3.2 and a subgradient $g \in \partial V(x)$. Then, we rely on the theory of linear programming with oracle [24], working in the *Turing model* of computation (a.k.a. *bit model*). In particular, all the computations are carried out with rational numbers. We now provide the proofs. subsection 5.1 deals with exact models whereas subsection 5.2 allows arbitrary probability distributions thanks to the use of approximate oracles.

5.1. Multistage programming with exact oracles. Recall that a polyhedron can be given in two manners. The "H-representation" provides an external description of the polyhedron, as the intersection of finitely many half-spaces. The "V-representation" provides an internal representation, writing the polyhedron as a Minkowski sum of a polytope (given as the convex hull of finitely many points) and of a polyhedral cone (generated by finitely many vectors).

We say that a polyhedron is *rational* if the inequalities in its H-representation are rational or, equivalently, the generators of its V-representation have rational coefficients. We shall say that a (convex) polyhedral function V is *rational* if its epigraph is a rational polyhedron.

Recall that, in the Turing model, the *size* (or encoding length see [24, 1.3]) of an integer $k \in \mathbb{Z}$ is $\langle k \rangle := 1 + \lceil \log_2(|k| + 1) \rceil$; the size of a rational $r = \frac{p}{q} \in \mathbb{Q}$ with p and q coprime integers, is $\langle r \rangle := \langle p \rangle + \langle q \rangle$. The size of a rational matrix or a vector, still denoted by $\langle \cdot \rangle$, is the sum of the sizes of its entries. The size of an inequality $\alpha^\top x \leqslant \beta$ is $\langle \alpha \rangle + \langle \beta \rangle$. The size of a H-representation of a polyhedron is the sum of the sizes of its inequalities and the size of a V-representation of a polyhedron is the sum of the sizes of its generators.

If the dimension of the ambient space is *fixed*, one can pass from one representation to the other one in *polynomial time*. Indeed, the double description algorithm allows one to get a V-representation from a H-representation, see the discussion at the end of section 3.1 in [21], and use McMullen's upper bound theorem ([39] and [24, 6.2.4]) to show that the computation time is polynomially bounded in the size of the H-representation. A fortiori, the size of the V-representation is polynomially bounded in the size of the H-representation. Dually, the same method allows one to obtain a H-representation from a V-representation. Hence, in the sequel, we shall use the term *size* of a polyhedron for the size of a V or H-representation: when dealing with polynomial-time complexity results in fixed dimension, whichever representation is used is irrelevant. In particular, we define the *size* $\langle N \rangle$ of a rational cone N as the size of a H or V representation of N.

We first observe that the size of the scenario tree arising in the exact quantization result becomes polynomial when suitable dimensions are fixed.

Proposition 5.1. *Let* $t \in \{2, \dots, T\}$*, and suppose that the dimensions* $n_t, \dots, n_T$ *and the cardinals* $\sharp(\operatorname{supp} \boldsymbol{\xi}_t), \dots, \sharp(\operatorname{supp} \boldsymbol{\xi}_T)$ *are fixed. Let* $\mathcal{T}$ *be the scenario tree constructed in Corollary 4.7. Then, the subtree of* $\mathcal{T}$ *rooted at an arbitrary node of depth* t *can be computed in polynomial time in* $\sum_{s=t}^{T} \sum_{\xi \in \operatorname{supp}(\boldsymbol{\xi}_s)} \langle \xi \rangle$.

Proof. Recall that a node of depth t of $\mathcal{T}$ is labeled by a sequence $(N_\tau, \xi_\tau)_{t_0 < \tau \leqslant t}$, where N_τ describes $\mathcal{N}_{t,\xi} = \bigwedge_{\sigma \in \mathcal{P}_{t,\xi}} -\mathcal{N}_{t,\xi,\sigma}$, where $\mathcal{P}_{t,\xi}$ is defined in (4.5a) by $\mathcal{P}_{t,\xi} := \mathcal{C}(\mathbb{R}^{n_t} \times \mathcal{P}_{t+1} \wedge \mathcal{F}\big(P_t(\xi)\big), \pi_{x_{t-1}}^{x_{t-1}, x_t})$, and $\mathcal{P}_{t+1} = \bigwedge_{\xi \in \operatorname{supp} \boldsymbol{\xi}_{t+1}} \mathcal{P}_{t+1,\xi}$.

Assume by induction that $\mathcal{P}_{t+1}$ and the subtrees of $\mathcal{T}$ rooted at a node of depth $t+1$ can be computed in polynomial time in $\sum_{s=t+1}^{T} \sum_{\xi \in \operatorname{supp}(\boldsymbol{\xi}_s)} \langle \xi \rangle$. Then $\sharp \mathcal{P}_{t+1}$ is polynomial in $\sum_{s=t+1}^{T} \sum_{\xi \in \operatorname{supp}(\boldsymbol{\xi}_s)} \langle \xi \rangle$. It is well known that (see [55, 3.9]) the number of chambers of a chamber complex $\mathcal{C}(\mathcal{Q}, \pi)$ is polynomial in $\langle \mathcal{Q} \rangle$ when both dimensions are fixed. Thus, for each $\xi \in \operatorname{supp}(\boldsymbol{\xi}_t)$ $\sharp \mathcal{P}_{t,\xi}$ is polynomial in $\langle \xi \rangle + \langle \mathcal{P}_{t+1} \rangle$ and thus in $\sum_{s=t}^{T} \sum_{\xi \in \operatorname{supp}(\boldsymbol{\xi}_s)} \langle \xi \rangle$ and we can compute the (maximal) chambers of the complexes $\mathcal{P}_{t,\xi}$ thanks to the algorithm in [10, 3.2] in polynomial time.

For each chamber σ of $\mathcal{P}_{t,\xi}$, thanks to a linear program, we find $x \in \operatorname{ri}(\xi)$ in polynomial time. The number of cones in $\mathcal{N}_{t,\xi,\sigma} = \mathcal{N}(P_t(\xi)_x)$ is equal to the number of faces of the fiber $P_t(\xi)_x$ which is polynomially bounded in the number of constraints $q \leqslant \langle \xi \rangle$ when the dimension n_t is fixed. Indeed, the McMullen upper-bound theorem [39], in its dual version, guarantees that a polytope of dimension m with f facets has $O(f^{\lfloor m/2 \rfloor})$ faces, see [47]. Thus, $\sharp \mathcal{N}_{t,\xi,\sigma}$ is polynomial in $\langle \xi_t \rangle$. By taking the common refinements, we can construct, in polynomial time, the nodes of $\mathcal{T}$ of depth t.

We recall the theory of linear programming with oracle applies to the class of "well described" polyhedra which are rational polyhedra with an a priori bound on the bit-sizes of the inequalities defining their facets, we refer the reader to [24] for a more detailed discussion of the notions (oracles) and results used here.

Definition 5.2 (first-order oracle). *Let* f *be a rational polyhedral function. We say that* f *admits a polynomial time (exact)* first-order oracle*, if there exists an oracle that takes as input a vector* x *and either returns a hyperplane separating* x *from* $\operatorname{dom}(f)$ *if* $x \notin \operatorname{dom}(f)$ *or returns* $f(x)$ *and* $g \in \partial V(x)$ *if* $x \in \operatorname{dom}(f)$*, in polynomial time in* $\langle x \rangle$.

Lemma 5.3. *Let* $Q \subset \mathbb{R}^d$ *be a polyhedron,* $c \in \mathbb{R}^d$ *a cost vector and* f *be a polyhedral function given by a first-order oracle. Futhermore, assume* $\operatorname{epi}(f)$ *and* Q *are well described. Then, the problem* $\min_{x \in Q} \ c^\top x + f(x)$ *can be solved in oracle-polynomial time in* $\langle c \rangle + \langle \operatorname{epi}(f) \rangle + \langle Q \rangle$.

Proof. The proof follows from the analysis of the ellipsoid method by Grötschel, Lovász and Schrijver. More precisely, the case where $\operatorname{dom}(f) = \mathbb{R}^d$ is tackled in Theorem 6.5.19 in [24] which shows that minimizing a polyhedral function with a well described epigraph over $\mathbb{R}^d$ can be done in polynomial time. If f has a general domain, we can write $f = \widetilde{f} + \mathbb{I}_{\operatorname{dom} f}$ where $\widetilde{f}$ is a polyhedral function with a well described epigraph and such that $\operatorname{dom} \widetilde{f} = \mathbb{R}^d$. E.g., we may obtain such an $\widetilde{f}$ by considering the inf-convolution of f with the polyhedral function $L\|\cdot\|_\infty$ where $L > 0$ is the Lipschitz constant of the restriction of f to its domain, with respect to the sup-norm, meaning that $|f(x) - f(y)| \leqslant L\|x - y\|_\infty$ for all $x, y \in \operatorname{dom} f$ and that L is the smallest constant with this property. Then, it is immediate to see that $\widetilde{f}$ coincides with f on $\operatorname{dom} f$ and that it is everywhere finite. Moreover, $\widetilde{f}$ is still well-described.

Then, noting that $\mathrm{epi}(f) = \mathrm{epi}(\widetilde{f}) \cap (\mathrm{dom}(f) \times \mathbb{R})$, we can adapt the proof of Theorem 6.5.19, *ibid.*, using Exercise 6.5.18 in this reference, which states that the intersection of well described polyhedra is well described. ∎

We do not require the distribution of the cost $\boldsymbol{c}$ to be described extensively. We only need to assume the existence of the following oracle.

DEFINITION 5.4 (cone-valuation oracle). *Let $\boldsymbol{c} \in L^1(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{R}^m)$ be an integrable cost distribution such that, for every* rational *cone N, the quantized probability $\check{p}_N$ and quantized cost $\check{c}_N$ are* rational. *We say that $\boldsymbol{c}$ admits a polynomial time (exact)* cone-valuation oracle*, if there exists an oracle which takes as input a rational polyhedral cone N and returns $\check{p}_N$ and $\check{c}_N$ in* polynomial time *in $\langle N \rangle$.*

THEOREM 5.5 (Cone valuation to first-order oracle). *Consider the value functions of MSLP defined in (1.2) . Assume that $T, n_2, \dots, n_T$, $\sharp(\mathrm{supp}\,\boldsymbol{\xi}_2), \cdots, \sharp(\mathrm{supp}\,\boldsymbol{\xi}_T)$ are fixed integers, and that $(\boldsymbol{c}_t, \boldsymbol{\xi}_t)_{2 \leqslant t \leqslant T}$ satisfies Assumption 2. Assume in addition that, every vector $\xi \in \mathrm{supp}(\boldsymbol{\xi}_t)$ has rational entries and that the probabilities $p_{t,\xi} := \mathbb{P}\big[\boldsymbol{\xi}_t = \xi\big]$ are rational numbers. Assume finally that every random variable $\boldsymbol{c}_t$ conditionally to $\{\boldsymbol{\xi}_t = \xi\}$, denoted by $\boldsymbol{c}_{t,\xi}$, admits a polynomial-time cone-valuation oracle (see Definition 5.4).*

Then, for all $t \geqslant 2$, V_t admits a polynomial time first-order oracle.

Proof. We start with the 2-stage case with deterministic constraints. We recall our notation $V(x) := \mathbb{E}\big[\min_{y \in \mathbb{R}^m} \boldsymbol{c}^\top y + \mathbb{I}_{Ay+Bx \leqslant b}\big]$. Let $x \in \mathbb{R}^n$ be an input vector. We first check if $x \in \pi(P) = \mathrm{dom}(V)$. By solving the dual of $\min_{y \in \mathbb{R}^m} \{\, 0 \mid Ay \leqslant b - Bx \}$, we either find an unbounded ray generated by $\lambda \in \mathbb{R}^q$ such that $\lambda \geqslant 0$, $\lambda^\top A = 0$ and $\lambda^\top(b - Bx) < 0$ or a $y \in \mathbb{R}^m$ such that $Ay \leqslant b - Bx$, so that $x \in \pi(P)$. In the former case we have $x \notin \pi(P)$, and we get a cut $\{x' \in \mathbb{R}^n \mid \lambda^\top B x' = \frac{\lambda^\top b + \lambda^\top A x}{2}\}$, separating $\pi(P) = \mathrm{dom}(V)$ from x.

So, we now assume that $x \in \pi(P)$, *i.e.*, $V(x) < +\infty$. We next show that we can compute $V(x)$ and a subgradient $\alpha \in \partial V(x)$ in polynomial time. Indeed, the McMullen upper-bound theorem [39], in its dual version, guarantees that a polytope of dimension m with f facets has $O(f^{\lfloor m/2 \rfloor})$ faces, see [47]. Since the number of cones in $\mathcal{N}(P_x)$ is equal to the number of faces of P_x which is polynomially bounded in the number of constraints $q \leqslant \langle \xi \rangle$, $\sharp\mathcal{N}(P_x)$ is polynomial in $\langle \xi \rangle$. Thus, since $\boldsymbol{c}$ is given by a cone valuation oracle, we can compute in polynomial time the collection of all quantized costs and probabilities $\check{c}_N$ and $\check{p}_N$, indexed by $N \in -\mathcal{N}(P_x)$. Then, by Theorem 3.2, we can compute $V(x)$ by solving a linear program for each cone $N \in -\mathcal{N}(P_x)$. Similarly, Theorem 3.6 allows us to compute a subgradient $\alpha \in \partial V(x)$. All these operations take a polynomial time.

The case of finitely supported stochastic constraints reduces to the case of deterministic constraints dealt with above, using $\mathrm{dom}(V) = \cap_{\xi \in \mathrm{supp}\,\boldsymbol{\xi}} \pi(P(\xi))$ and $V(x) = \sum_{\xi \in \mathrm{supp}\,\boldsymbol{\xi}} p_\xi \widetilde{V}(x|\xi)$ where $\widetilde{V}(x|\xi) := \mathbb{E}\big[\hat{V}(x, \boldsymbol{c}, \boldsymbol{\xi}) \mid \boldsymbol{\xi} = \xi\big]$.

We finally deal with the multistage case in a similar way, using the quantization result Corollary 4.7 in extensive form. Applying Proposition 5.1, the quantized costs and probabilities arising there can be computed by a polynomial number of calls to the cone-valuation oracle. This provides a first order oracle for the expected cost-to-go function V_t. ∎

We now refine the definition of cone-valuation oracle, to take into account situations in which the distribution of the random cost $\boldsymbol{c}$ is specified by a parametric model. We shall say that such a distribution admits a polynomial-time *parametric cone-valuation oracle* if there is an oracle that takes as input the parameters of the distribution, together with a rational cone N, and outputs the quantized probability $\check{p}_N$ and cost $\check{c}_N$. Especially, we consider the following situations:

1. *Deterministic distribution* equal to a rational cost c. We set $\langle \boldsymbol{c} \rangle := \langle c \rangle$
2. *Exponential distribution on a rational cone* K with rational parameter θ. We set $\langle \boldsymbol{c} \rangle := \langle K \rangle + \langle \theta \rangle$
3. *Uniform distribution on a rational polyhedron* Q such that $\mathrm{Aff}(Q) = \{y \in \mathbb{R}^m \mid \forall j \in J \subset [m], y_j = q_j \in \mathbb{Q}\}$ where J is a subset of $[m]$ and q_j are rational numbers (in particular, Q is full dimensional when $J = \emptyset$). We set: $\langle \boldsymbol{c} \rangle = \langle Q \rangle$
4. *Mixtures of the above distributions*, *i.e.*, convex combination with rational coefficients $(\lambda^k)_{k\in[l]}$ of distributions of random variables $(\boldsymbol{c}_k)_{k\in[l]}$ satisfying 1. 2. or 3. Then, we set $\langle \boldsymbol{c} \rangle = \sum_{k=1}^{l} \langle \boldsymbol{c}_k \rangle + \langle \lambda_k \rangle$.

THEOREM 5.6. *Assume that the dimension m is fixed, and that $\boldsymbol{c}$ is distributed according to any of the above laws (deterministic, exponential, uniform, or mixture). Then, the random cost $\boldsymbol{c}$ admits a polynomial-time parametric cone-valuation oracle.*

Proof. 1. *Case of a deterministic distribution.* We first check whether $c \in \mathrm{ri}(N)$, which can be done in polynomial time, see section 6.5 of [24]. Then, if $c \in \mathrm{ri}(N)$, we set $\check{c}_N = c$ and $\check{p}_N = 1$ otherwise $\check{c}_N = 0$ and $\check{p}_N = 0$.

2. *Case of an exponential distribution.* Since the dimension is fixed, for every polyhedron R, we can triangulate $R \cap \mathrm{supp}(\boldsymbol{c})$ and partition it into (relatively open) simplices and simplicial cones $(S_k)_{k\in[l]}$, and by Stanley upper bound theorem, the size l of the triangulation is polynomial in $\langle R \rangle$. By using the exponential valuation of a simplicial cone in Table 1 see also [1, (8.2.2)] or [9] , we compute in polynomial time $\check{p}_R = \sum_{k=1}^{l} \check{p}_{S_k}$ and $\check{c}_R = \sum_{k=1}^{l} \check{p}_{S_k} \check{c}_{S_k} / \check{p}_R$ if $\check{p}_R = 0$ and $\check{c}_R = 0$ otherwise.

3. *Case of a uniform distribution.* After triangulating (as in the case of an exponential distribution), we may suppose that the support of the distribution is a simplex S, so that $Q = S$. If this simplex S is full dimensional, then its volume is given by a determinantal expression, and so, it is rational (see *e.g.,* [23] 3.1). Then, the formulas of Table 1 yield the result. If this simplex is not full dimensional, we have $\mathrm{Aff}(S) = \{y \in \mathbb{R}^m \mid \forall j \in J, y_j = q_j\}$, a similar formula holds, ignoring the coordinates of y whose indices are in the set J.

4. *Case of mixtures of distributions.* Trivial reduction to the previous cases. ∎

REMARK 5.7. *The conclusion of Theorem 5.6 does not carry over to the uniform distribution on a general polytope of dimension $k < n$. The condition that $\mathrm{Aff}(Q) = \{y \in \mathbb{R}^m \mid \forall j \in J, y_j = q_j\}$ ensures that the orthogonal projection on $\mathrm{Aff}(Q)$ preserves rationality, which entails that the k-dimensional volume of Q is a rational number. In general, this volume is obtained by applying the Cayley Menger determinant formula (see for example [23, 3.6.1]), and it belongs to a quadratic extension of the field of rational numbers. For example, if Δ_d is the canonical simplex $\{\lambda \in \mathbb{R}_+^{d+1} | \sum_{i=1}^{d+1} \lambda_i = 1\}$ then $\mathrm{Vol}(\Delta_d) = \frac{\sqrt{d+1}}{d!}$.*

For the Gaussian distribution, $\check{c}_S$ and $\check{p}_S$ can be determined in terms of solid angles (see [45]) arising in Table 1. These coefficients are generally involving the number π and Euler's Γ function, and thus they are irrational.

COROLLARY 5.8 (MSLP is polynomial for fixed dimensions). *Consider the problem (1.1) . Assume that $T, n_2, \dots, n_T$, $\sharp(\mathrm{supp}\, \boldsymbol{\xi}_2)$, $\cdots$,$\sharp(\mathrm{supp}\, \boldsymbol{\xi}_T)$ are fixed integers, that $(\boldsymbol{c}_t, \boldsymbol{\xi}_t)_{2\leqslant t\leqslant T}$ satisfies Assumption 2. Suppose in addition that, for all $\xi \in \mathrm{supp}(\boldsymbol{\xi}_t)$, $p_{t,\xi} := \mathbb{P}\big[\boldsymbol{\xi}_t = \xi\big]$ and ξ are rational and that the random variable*

$\boldsymbol{c}_t$ *conditionally to* $\{\boldsymbol{\xi}_t = \xi\}$, *denoted by* $\boldsymbol{c}_{t,\xi}$, *is of the type considered in* Theorem 5.6.

Then, Problem (1.1) *can be solved in a time that is polynomial in the input size* $\langle c_1\rangle + \langle \xi_1\rangle + \sum_{t=2}^{T}\sum_{\xi\in\operatorname{supp}(\boldsymbol{\xi}_t)}(\langle \boldsymbol{c}_{t,\xi}\rangle + \langle\xi\rangle + \langle p_{t,\xi}\rangle)$.

Proof. We first show by backward induction that the epigraph $\operatorname{epi}(V_2)$ is well described. The dynamic programming equation (1.2) allows us to compute a H-representation of $\operatorname{epi}(V_t)$ from a H-representation of $\operatorname{epi}(V_{t+1})$. Indeed, by Theorem 4.6, we have

$$V_t(x_{t-1}) = \sum_{\xi\in\operatorname{supp}(\boldsymbol{\xi}_t)} p_{t,\xi} \sum_{N\in\mathcal{N}_{t,\xi}} \check{p}_{t,N|\xi} \min_{x_t\in\mathbb{R}^{n_t}} Q_{t,N|\xi}(x_t, x_{t-1}) \ , \text{ with}$$

$$Q_{t,N|\xi}(x_t, x_{t-1}) := \check{c}_{t,N|\xi}^\top x_t + V_{t+1}(x_t) + \mathbb{I}_{(x_t,x_{t-1})\in P_t(\xi)} \ .$$

We then have

$$\operatorname{epi}(Q_{t,N|\xi}) = \big(\operatorname{epi}(x_t \mapsto \check{c}_{t,N|\xi}^\top x_t) + \operatorname{epi}(V_{t+1})\big) \ \cap \ (P_t(\xi)\times\mathbb{R}),$$

$$\operatorname{epi}(V_t) = \sum_{\xi\in\operatorname{supp}(\boldsymbol{\xi}_t)} p_{t,\xi} \sum_{N\in\mathcal{N}_{t,\xi}} \check{p}_{t,N|\xi}\ \pi^{x_{t-1},x_t,z}_{x_{t-1},z}\big(\operatorname{epi}(Q_{t,N|\xi})\big) \ ,$$

recalling that $\pi^{x_{t-1},x_t,z}_{x_{t-1},z}$ denotes the projection mapping $(x_{t-1}, x_t, z) \mapsto (x_{t-1}, z)$. Well described polyhedra are stable under the operations of projection, intersection, and Minkowski sum, see in particular [24, 6.5.18]. It follows that $\operatorname{epi}(V_t)$ is well described. Then, the corollary follows from Lemma 5.3, Theorem 5.5 and Theorem 5.6. ∎

5.2. Multistage programming with approximate oracles. We finally consider the situation in which the law of the cost distribution is only known approximately. Hence, we relax the notion of cone-valuation oracle, as follows.

Definition 5.9 (Weak cone-valuation oracle). *Let* $\boldsymbol{c}\in L(\Omega,\mathcal{A},\mathbb{P},\mathbb{R}^m)$ *be an integrable cost distribution. We say that* $\boldsymbol{c}$ *admits a polynomial time* weak cone-valuation oracle, *if there exists an oracle which takes as input a rational polyhedral cone* N *together with a rational number* $\varepsilon > 0$, *and returns a rational number* $\widetilde{p}_N$ *and a rational vector* $\widetilde{c}_N$ *such that* $|\widetilde{p}_N - \check{p}_N| \leqslant \varepsilon$ *and* $\|\widetilde{c}_N - \check{c}_N\| \leqslant \varepsilon$, *in a time that is* polynomial *in* $\langle N\rangle + \langle\varepsilon\rangle$.

Definition 5.10 (Weak first-order oracle). *Let* f *be a rational polyhedral function. We say that* f *admits a polynomial time* weak first-order oracle, *if there exists an oracle that takes as input a vector* x *and either returns a hyperplane separating* x *from* $\operatorname{dom}(f)$ *if* $x\notin\operatorname{dom}(f)$ *or returns a scalar* $\widetilde{f}$ *and a vector* $\widetilde{g}$ *such that* $|\widetilde{f} - f(x)| \leqslant \varepsilon$ *and* $d\big(\widetilde{g}, \partial f(x)\big) \leqslant \varepsilon$ *if* $x\in\operatorname{dom}(f)$, *in a time which is polynomial in* $\langle x\rangle + \langle\varepsilon\rangle$.

Remark 5.11. *In our definition of weak first order oracle, we require that feasibility (*$x\in\operatorname{dom}(f)$*) be tested exactly, whereas the value and a subgradient of the function are only given approximately. This is suitable to the present setting, in which the main difficulty resides in the approximation of the function (which may take irrational values for relevant cost distributions).*

We now rely on the theory of linear programming with weak separation oracles developed in [24]. Let $C\subset\mathbb{R}^d$ be convex set, for $\varepsilon > 0$, let $S(C,\varepsilon) := \{x\in\mathbb{R}^d \mid \|x-y\| \leqslant \varepsilon\}$ and $S(C,-\varepsilon) := \{x\in\mathbb{R}^d \mid B(x,\varepsilon)\subset C\}$ where $B(x,\varepsilon)$ denotes the Euclidean ball centered at x of radius ε. A *weak separation oracle* for a convex set $C\subset\mathbb{R}^d$ takes as argument a vector $x\in\mathbb{R}^d$ and a rational number $\varepsilon > 0$, and either asserts that $x\in S(C,\varepsilon)$ or returns a rational vector $\gamma\in\mathbb{R}^d$, of norm one, and a rational scalar δ, such that $\gamma^\top y \leqslant \gamma^\top x + \varepsilon$ for all $y\in S(C,-\varepsilon)$.

Theorem 5.12 (Weak cone valuation to weak first-order oracle). *Consider the value functions of problem* (1.1) *defined in* (1.2) *. Assume that* $T, n_2, \dots, n_T$*,* $\sharp(\operatorname{supp} \boldsymbol{\xi}_2)$*,* $\cdots$*,*$\sharp(\operatorname{supp} \boldsymbol{\xi}_T)$ *are fixed integers, and that* $(\boldsymbol{c}_t, \boldsymbol{\xi}_t)_{2 \leqslant t \leqslant T}$ *satisfies Assumption* 2*. Assume in addition that, every vector* $\xi \in \operatorname{supp}(\boldsymbol{\xi}_t)$ *has rational entries and that the probabilities* $p_{t,\xi} := \mathbb{P}\big[\boldsymbol{\xi}_t = \xi\big]$ *are rational numbers. Assume finally that the diameters of* $\operatorname{dom} V_t$*, for* $t \geqslant 2$*, are bounded by a rational constant* R*, and that every random variable* $\boldsymbol{c}_t$ *conditionally to* $\{\boldsymbol{\xi}_t = \xi\}$*, denoted by* $\boldsymbol{c}_{t,\xi}$*, admits a polynomial-time weak cone-valuation oracle (see Definition* 5.4*).*

Then, for all $t \geqslant 2$*,* V_t *admits a polynomial time weak first-order oracle.*

Proof. The proof is similar to the one of Theorem 5.5. The main difference is that we need an a priori bound R on the diameter of $\operatorname{dom} V_t$, so that if $d(\widetilde{g}, \partial V_t(x)) \leqslant \varepsilon$, then, using Cauchy-Schwarz inequality, $V_t(y) - V_t(x) \geqslant \widetilde{g} \cdot (y - x) - \varepsilon R$ holds for all $y \in \operatorname{dom} V_t$. Together with and approximation of $V_t(x)$, this allows us to get a weak separation oracle for the epigraph of V_t. ∎

Corollary 5.13 (Approximate (MSLP) is polynomial-time for fixed recourse dimension m). *Consider Problem* (1.1)*. Let* $T, n_2, \dots, n_T$*,* $\sharp(\operatorname{supp} \boldsymbol{\xi}_2)$*,* $\cdots$*,*$\sharp(\operatorname{supp} \boldsymbol{\xi}_T)$ *be fixed integers. Assume finally that the diameters of* $\operatorname{dom} V_t$*, for* $t \geqslant 2$*, are bounded by* $R \in \mathbb{Q}$*, and that for all* $\xi \in \operatorname{supp}(\boldsymbol{\xi}_t)$*, the random variable* $\boldsymbol{c}_t$ *conditionally to* $\{\boldsymbol{\xi}_t = \xi\}$*, denoted by* $\boldsymbol{c}_{t,\xi}$*, admits a polynomial-time weak cone-valuation oracle.*

Then, there exists an algorithm that either asserts that Problem (1.1) *is infeasible or find a feasible solution* x^* *whose cost does not exceed the cost of an optimal solution by more than* ε*, in polynomial-time in* $\langle \varepsilon \rangle + \langle c_1 \rangle + \langle \xi_1 \rangle + \sum_{t=2}^{T} \sum_{\xi \in \operatorname{supp}(\boldsymbol{\xi}_t)} (\langle \boldsymbol{c}_{t,\xi} \rangle + \langle \xi \rangle + \langle p_{t,\xi} \rangle) + \langle R \rangle$*. In particular, its complexity is polynomial in* $\log(1/\varepsilon)$*.*

Proof. This follows from Theorem 5.12, using the result analogous to Lemma 5.3 for weak separation oracles, see [24, 6.5.19]. ∎

Finally, we show that every absolutely continuous cost distribution, with a suitable density function, admits a polynomial-time weak cone-valuation oracle.

Definition 5.14. *A density function* $f : \mathbb{R}^n \to \mathbb{R}_+$ *is* combinatorially tight *if:*

1. *there is a polynomial time algorithm which, given a rational number* $\varepsilon > 0$*, returns a rational number* $r > 0$ *such that* $\int_{\|x\| > r} f(x) dx \leqslant \varepsilon$*.*
2. *there is a polynomial time algorithm, which given a rational vector* $x \in \mathbb{R}^n$*, and a rational number* $\varepsilon > 0$*, returns an* ε *approximation of* $f(x)$*.*

The terminology is inspired by the notion of tightness from measure theory (analogous to condition 1 in Definition 5.14).

We shall need a classical result on the numerical approximation of multidimensional integrals. The *total variation in the sense of Hardy and Krause*, $\|f\|_{\mathrm{BVHK}}$, of a function f on a n dimensional hypercube is defined in [11, Def. p.352]). In particular, if f is of regularity class $\mathcal{C}^n$, $\|f\|_{\mathrm{BVHK}}$ is finite. The error made when approximating the integral of a function of n variables by its Riemann sum taken on a regular grid with k points is bounded by $(n\|f\|_{\mathrm{BVHK}})/k^{1/n}$, see [11, p.352].

Proposition 5.15. *Suppose that a cost distribution* $\boldsymbol{c}$ *admits a density function* $f : \mathbb{R}^n \to \mathbb{R}_+$*, that is such that the function* $(1 + \|\cdot\|) f$ *is combinatorially tight and that it has a finite total variation in the sense of Hardy and Krause, bounded by an a priori constant. Suppose that the dimension* n *is fixed. Then,* $\boldsymbol{c}$ *admits a polynomial-time weak cone valuation oracle.*

Proof. Given a rational cone N, we need to approximate the integrals $\int_N f(c) dc$ and $\int_N c f(c) dc$, up to the precision ε. Using the tightness condition, it suffices to approximate the integrals of the same functions restricted to the domain $N_r := N \cap B_\infty(0, r)$, where $B_\infty(0, r)$ denotes the sup-norm ball of radius r, and the encoding length of r is polynomially bounded in the encoding length of ε. We only discuss the approximation of $\int_{N_r} cf(c)dc$ (the case of $\int_{N_r} f(c)dc$ being simpler). We denote by $\widetilde{c}_{N_r}$ the approximation of $\int_{N_r} cf(c)dc$ provided by taking the Riemann sum of the function $c \mapsto cf(c)$ over the grid $([-r, r))^n \cap ((r/M)\mathbb{Z})^n$, which has $(2M)^r$ points. Then, setting $g := (1 + \| \cdot \|)f$, it follows from [11, Th. p 352] recalled above that $\| \int_{N_r} cf(c)dc - \widetilde{c}_{N_r} \| \leqslant n \|g\|_{\mathrm{BVHK}}/(2M)$. Hence, for a fixed dimension n, we can get an ε approximation of $\int_N cf(c)dc$ in a time polynomial in the encoding length of ε. ☐

REMARK 5.16. *Proposition 5.15 and Corollary 5.13 entail that, under the previous fixed-parameter restrictions (including dimensions of the recourse spaces), the MSLP problem is polynomial-time approximately solvable for a large class of cost distributions. This applies in particular to distributions like Gaussians, which are combinatorially tight. In this case, condition 1 of Definition 5.14, whereas condition 2 follows from the result of [8], implying that the exponential function, restricted to the interval* $(-\infty, 0]$*, can be approximated in polynomial time.*

6. Conclusion and perspectives. This polyhedral approach enlightens the structure of multistage stochastic linear problems. It allows us to derive theoretical complexity results for a large class of random variables. However, the combinatorics of the polyhedral used suffers from the curse of dimensionality and all chamber complexes and normal fans cannot be computed in practice in high dimension. To avoid this problem, we leverage in [17] the local exact quantization result to define generalized adaptive partition based algorithms for 2SLP when the constraints have general distributions. This technique can be adapted to the multistage setting, see [18]. Moreover, we exploit the present approach to develop, in [19], a "higher order" simplex algorithm, following a path on the vertices of the chamber complex, and updating locally the normal fan. Finally, these new objects, and in particular the weighted fiber polyhedron may allow us to better understand the dependence of MSLP with the distribution of random variables, for example by linking it with the nested distance [41], in order to improve the results on scenario tree approximations, whether they are statistical or not.

References.

[1] Alexander Barvinok. *Integer points in polyhedra*, volume 452. European Mathematical Society, 2008.

[2] Louis J. Billera, Israel M Gelfand, and Bernd Sturmfels. Duality and minors of secondary polyhedra. *Journal of Combinatorial Theory, Series B*, 57(2):258–268, 1993.

[3] Louis J. Billera and Bernd Sturmfels. Fiber polytopes. *Annals of Mathematics*, pages 527–549, 1992.

[4] John R. Birge. Decomposition and partitioning methods for multistage stochastic linear programs. *Operations research*, 33(5):989–1007, 1985.

[5] John R. Birge and Francois Louveaux. *Introduction to stochastic programming.* Springer Science & Business Media, 2011.

[6] Alexander Black, Jesús De Loera, Sean Kafer, and Laura Sanità. On the simplex method for 0/1 polytopes, 2021.

[7] Alexander E Black, Jesús A De Loera, Niklas Lütjeharms, and Raman Sanyal. The polyhedral geometry of pivot rules and monotone paths. *SIAM Journal on Applied Algebra and Geometry*, 7(3):623–650, 2023.

[8] J. M. Borwein and P. B. Borwein. On the complexity of familiar functions and numbers. *SIAM Review*, 30(4):589–601, 1988.

[9] Michel Brion. Points entiers dans les polyèdres convexes. In *Annales scientifiques de l'Ecole Normale Supérieure*, volume 21, pages 653–663, 1988.

[10] Philippe Clauss and Vincent Loechner. Parametric analysis of polyhedral iteration spaces. *Journal of VLSI signal processing systems for signal, image and video technology*, 19(2):179–194, 1998.

[11] Ph J. Davis and Ph. Rabinowitz. *Methods of numerical integration. 1984*. Comput. Sci. Appl. Math. Acad. Press, 1984.

[12] Jesús A. De Loera, Raymond Hemmecke, and Matthias Köppe. *Algebraic and geometric ideas in the theory of discrete optimization*. SIAM, 2012.

[13] Jesús A. De Loera, Jörg Rambau, and Francisco Santos. *Triangulations Structures for algorithms and applications*. Springer, 2010.

[14] Jitka Dupačová, Nicole Gröwe-Kuska, and Werner Römisch. Scenario reduction in stochastic programming. *Mathematical Programming*, 95(3):493–511, 2003.

[15] Martin Dyer and Leen Stougie. Computational complexity of stochastic programming problems. *Mathematical Programming*, 106(3):423–432, 2006.

[16] Nalin C.P. Edirisinghe and William T. Ziemba. Bounds for two-stage stochastic programs with fixed recourse. *Mathematics of Operations Research*, 19(2):292–313, 1994.

[17] Maël Forcier and Vincent Leclère. Generalized adaptive partition-based method for two-stage stochastic linear programs: geometric oracle and analysis. *Operations Research Letters*, 2022.

[18] Maël Forcier and Vincent Leclère. Convergence of trajectory following dynamic programming algorithms for multistage stochastic problems without finite support assumptions. *Journal of Convex Analysis*, (3):951–999, 2023.

[19] Maël Forcier. *Multistage stochastic optimization and polyhedral geometry*. Phd manuscript, École des Ponts, 2022.

[20] Komei Fukuda. Lecture: Polyhedral computation, spring 2016. 2016.

[21] Komei Fukuda and Alain Prodon. Double description method revisited. In *Franco-Japanese and Franco-Chinese Conference on Combinatorics and Computer Science*, pages 91–111. Springer, 1995.

[22] Israel M Gelfand, Mikhail M Kapranov, and Andrei V Zelevinsky. *Discriminants, Resultants, and Multidimensional Determinants*. Springer, 1994.

[23] Peter Gritzmann and Victor Klee. On the complexity of some basic problems in computational convexity, ii. volume and mixed volumes. In *Polytopes: Abstract, Convex and Computational*, pages 373–466. Springer, 1994.

[24] Martin Grötschel, László Lovász, and Alexander Schrijver. *Geometric algorithms and combinatorial optimization*, volume 2 of *Algorithms and Combinatorics*. Springer Science & Business Media, 2012.

[25] Branko Grünbaum. *Convex polytopes*, volume 221. Springer Science & Business Media, 2013.

[26] Grani A. Hanasusanto, Daniel Kuhn, and Wolfram Wiesemann. A comment on "computational complexity of stochastic programming problems". *Mathematical Programming*, 159(1-2):557–569, 2016.

[27] Holger Heitsch and Werner Römisch. Scenario reduction algorithms in stochastic programming. *Computational optimization and applications*, 24(2-3):187–206, 2003.

[28] Michal Kaut and Stein W. Wallace. Evaluation of scenario-generation methods for stochastic programming. *Pacific Journal of Optimization*, 3(2):257–271, 2007.

[29] Kim-Manuel Klein and Janina Reuter. Collapsing the tower-on the complexity of multistage stochastic ips∗. In *Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 348–358. SIAM, 2022.

[30] Daniel Kuhn. *Generalized bounds for convex multistage stochastic programs*, volume 548. Springer Science & Business Media, 2006.

[31] Guanghui Lan. Complexity of stochastic dual dynamic programming. *Mathematical Programming*, pages 1–38, 2020.

[32] Jean B. Lasserre. Simple formula for integration of polynomials on a simplex. *BIT Numerical Mathematics*, 61(2):523–533, 2021.

[33] Jim Lawrence. Polytope volume computation. *Mathematics of computation*, 57(195):259–271, 1991.

[34] Hendrik W. Lenstra Jr. Integer programming with a fixed number of variables. *Mathematics of operations research*, 8(4):538–548, 1983.

[35] Vincent Loechner and Doran K. Wilde. Parameterized polyhedra and their vertices. *International Journal of Parallel Programming*, 25(6):525–549, 1997.

[36] Shu Lu and Stephen M. Robinson. Normal fans of polyhedral convex sets. *Set-Valued Analysis*, 16(2-3):281–305, 2008.

[37] Francesca Maggioni, Elisabetta Allevi, and Marida Bertocchi. Bounds in multistage linear stochastic programming. *Journal of Optimization Theory and Applications*, 163(1):200–229, 2014.

[38] Léo Mathis and Chiara Meroni. Fiber convex bodies. *Discrete & Computational Geometry*, November 2022.

[39] Peter McMullen. The maximum numbers of faces of a convex polytope. *Mathematika*, 17(2):179–184, 1970.

[40] Mario V.F. Pereira and Leontina M.V.G. Pinto. Multi-stage stochastic optimization applied to energy planning. *Mathematical programming*, 52(1-3):359–375, 1991.

[41] Georg Ch. Pflug and Alois Pichler. A distance for multistage stochastic optimization models. *SIAM Journal on Optimization*, 22(1):1–23, 2012.

[42] András Prékopa. *Stochastic programming*, volume 324. Springer Science & Business Media, 2013.

[43] Jörg Rambau. *Polyhedral subdivisions and projections of polytopes*. Phd thesis, TU Berlin, 1996.

[44] Jörg Rambau and Günter M. Ziegler. Projections of polytopes and the generalized baues conjecture. *Discrete & Computational Geometry*, 16(3):215–237, 1996.

[45] Jason M. Ribando. Measuring solid angles beyond dimension three. *Discrete & Computational Geometry*, 36(3):479–487, 2006.

[46] R. Tyrrell Rockafellar and Roger J.-B. Wets. *Variational analysis*, volume 317. Springer Science & Business Media, 2009.

[47] R. Seidel. The upper bound theorem for polytopes: an easy proof of its asymptotic version. *Computational Geometry*, 5(2):115 – 116, 1995.

[48] Alexander Shapiro. Analysis of stochastic dual dynamic programming method. *European Journal of Operational Research*, 209(1):63–72, 2011.

[49] Alexander Shapiro, Darinka Dentcheva, and Andrzej Ruszczyński. *Lectures on stochastic programming: modeling and theory*. SIAM, 2014.

[50] Alexander Shapiro and Arkadi Nemirovski. On complexity of stochastic programming problems. In *Continuous optimization*, pages 111–146. Springer, 2005.

[51] David B. Shmoys and Chaitanya Swamy. An approximation scheme for stochastic linear programming and its application to stochastic integer programs. *Journal of the ACM (JACM)*, 53(6):978–1012, 2006.

[52] Richard P. Stanley. The upper bound conjecture and Cohen-Macaulay rings. *Studies in Applied Mathematics*, 54(2):135–142, 1975.

[53] Bernd Sturmfels and Rekha R. Thomas. Variation of cost functions in integer programming. *Mathematical Programming*, 77(2):357–387, 1997.

[54] Richard M. Van Slyke and Roger Wets. L-shaped linear programs with applications to optimal control and stochastic programming. *SIAM Journal on Applied Mathematics*, 17(4):638–663, 1969.

[55] Sven Verdoolaege, Kevin M. Woods, Maurice Bruynooghe, and Ronald Cools. Computation and manipulation of enumerators of integer projections of parametric polytopes. *CW Reports*, pages 104–104, 2005.

[56] David Walkup and Roger J.-B. Wets. Lifting projections of convex polyhedra. *Pacific Journal of Mathematics*, 28(2):465–475, 1969.

[57] Shixuan Zhang and Xu Andy Sun. Stochastic dual dynamic programming for multistage stochastic mixed-integer nonlinear optimization. *Mathematical Programming*, 196(1):935–985, 2022.

[58] Günter M. Ziegler. *Lectures on polytopes*, volume 152. Springer Science & Business Media, 2012.